Construction of bivariate symmetric orthonormal wavelets with short support

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In this paper, we give a parameterization of the class of bivariate symmetric orthonormal scaling functions with filter size 6×6 using the standard dilation matrix $2I$. In addition, we give two families of refinable functions which are not orthonormal but have associated tight frames. Finally, we show that the class of bivariate symmetric scaling functions with filter size 8×8 can not have two or more vanishing moments.

Key Words: bivariate, nonseparable, symmetric, wavelets, vanishing moments

1. INTRODUCTION

The most common wavelets used for image processing are the tensor-product of univariate compactly supported orthonormal wavelets. Of this class of wavelets, only the Haar wavelet is symmetric which gives its associated filter the property of linear phase. Since Daubechies' work[5], numerous generalizations of wavelets have been developed including biorthogonal wavelets, multiwavelets, and bivariate wavelets. Since 1992, several examples of bivariate compactly supported orthonormal and biorthogonal wavelets have been constructed. See Cohen and Daubechies'93 [3] for nonseparable bidimensional wavelets, J. Kovačević and M. Vetterli'92[8] for nonseparable filters and wavelets based on a generalized dilation matrix, He and Lai'97[6] for the complete solution of bivariate compactly supported wavelets with filter size up to 4×4, Belogay and Wang'99[2] for a special construction of bivariate nonseparable wavelets for any given regularity, and Ayache'99 [1] for nonseparable dyadic compactly supported wavelets with arbitrary regularity. See also

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Cohen and Schlenker'93[4], Riemenschneider and Shen'97[11], and He and Lai'98[7] for bivariate biorthogonal box spline wavelets.

It is well-known that in the univariate setting, there does not exist symmetric compactly supported orthonormal wavelets except Haar for dilation factor 2. We are interested in the construction of symmetric wavelets in the bivariate setting with dilation matrix $2I$ which have compact support and vanishing moments. We start with a scaling funtion ϕ. Let

$$\hat{\phi}(\omega_1, \omega_2) = \prod_{k=1}^{\infty} m(e^{\omega_1/2^k}, e^{\omega_2/2^k})$$

be the Fourier transform of ϕ, where

$$m(x, y) = \sum_{j=0}^{N} \sum_{k=0}^{N} c_{jk} x^j y^k$$

is a trigonometric polynomial satisfying $m(1,1) = 1$. In addition, the trigonometric polynomial $m(x, y)$ satisfies the orthonormality condition

$$|m(x,y)|^2 + |m(-x,y)|^2 + |m(x,-y)|^2 + |m(-x,-y)|^2 = 1.$$

Let $\psi_i(x, y)$ be the corresponding wavelet satisfying

$$\hat{\psi}_i(\omega_1, \omega_2) = m_i(e^{\omega_1/2}, e^{\omega_2/2})\hat{\phi}(\omega_1/2, \omega_2/2), i = 1, 2, 3,$$

where the m_i are trigonometric polynomials such that the following matrix

$$\begin{bmatrix} m(x,y) & m(-x,y) & m(x,-y) & m(-x,-y) \\ m_1(x,y) & m_1(-x,y) & m_1(x,-y) & m_1(-x,-y) \\ m_2(x,y) & m_2(-x,y) & m_2(x,-y) & m_2(-x,-y) \\ m_3(x,y) & m_3(-x,y) & m_3(x,-y) & m_3(-x,-y) \end{bmatrix}$$

is unitary. Moreover, we are interested in symmetric scaling functions ϕ with a certain number of vanishing moments in the sense that their associated trigonometric polynomial $m(x, y)$ satisfies

$$m(1/x, 1/y) = x^{-N} y^{-N} m(x, y)$$

as well as

$$\frac{\partial^k}{\partial x^k} m(x,y)\bigg|_{x=-1} = \frac{\partial^k}{\partial y^k} m(x,y)\bigg|_{y=-1} = 0, \qquad 0 \le k \le M-1.$$

The symmetry condition provides ϕ with the property of linear phase and the vanishing moment conditions provide ϕ with polynomial reproduction up to degree $M-1$. If $m(x, y)$ satisfies the symmetric property, then the associated wavelets can easily be found by using $\hat{\phi}$ and

$$\begin{aligned} m_1(x,y) &= m(-x,y) \\ m_2(x,y) &= x \cdot m(x,-y) \\ m_3(x,y) &= x \cdot m(-x,-y). \end{aligned}$$

In summary, we are looking for trigonometric polynomials $m(x,y) = \sum_{j=0}^{N}\sum_{k=0}^{N} c_{jk}x^j y^k$ which satisfy the following properties:

(i) Existence: $m(1,1) = 1$.

(ii) Orthogonality: $|m(x,y)|^2 + |m(-x,y)|^2 + |m(x,-y)|^2 + |m(-x,-y)|^2 = 1$.

(iii) Symmetry: $m(1/x,1/y) = x^{-N}y^{-N}m(x,y)$.

(iv) M vanishing moments: $m(x,y) = (x+1)^M(y+1)^M\tilde{m}(x,y)$ where $\tilde{m}(x,y)$ is another trigonometric polynomial.

In this paper, we construct a complete parameterization of all trigonometric polynomials $m(x,y)$ which satisfy the symmetry condition, the vanishing moment condition, and the orthonormality condition for $N=5$ and $M=1$. Within this class, we identify a two-parameter family which contains the trigonometric polynomials associated with scaling functions. Outside of this two-parameter family, we show that the remaining trigonometric functions are not associated with scaling functions but instead determine families of tight frames. Finally, we show that there are no trigonometric polynomials for $N=7$ and $M=2$, and consequently no symmetric bivariate scaling functions with two vanishing moments for the support size we are considering.

The paper is organized as follows. Section 2 gives the parameterized solution when $N=5$ and $M=1$. The problem is broken down into four cases which are dealt with in turn. Section 3 discusses the orthonormality of the solutions from Section 2 and concludes with a numerical experiment comparing Haar, D4, and one solution from Section 2. The last two sections show that these trigonometric polynomials cannot have higher vanishing moments(i.e $M \geq 2$) even for $N=7$.

2. THE 6 × 6 CASE

Our goal is to parameterize the coefficients of the trigonometric polynomials which satisfy properties (i)-(iv). We begin our investigation with trigonometric polynomials whose filter size is 6×6 with one vanishing moment, i.e. $N=5$ and $M=1$. The case $N=1$ is trivially the tensor product Haar function, and the case $N=3$ has eight singleton solutions given by He and Lai'97 [6].

Let us express $m(x,y)$ in its polyphase form, i.e.,

$$\begin{aligned} m(x,y) &= f_a(x^2,y^2) + xf_b(x^2,y^2) + yf_c(x^2,y^2) + xyf_d(x^2,y^2) \\ &= \begin{bmatrix} 1 \\ y \\ y^2 \\ y^3 \\ y^4 \\ y^5 \end{bmatrix}^T \begin{bmatrix} a_0 & b_0 & a_1 & b_1 & a_2 & b_2 \\ c_0 & d_0 & c_1 & d_1 & c_2 & d_2 \\ a_3 & b_3 & a_4 & b_4 & a_5 & b_5 \\ c_3 & d_3 & c_4 & d_4 & c_5 & d_5 \\ a_6 & b_6 & a_7 & b_7 & a_8 & b_8 \\ c_6 & d_6 & c_7 & d_7 & c_8 & d_8 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \\ x^3 \\ x^4 \\ x^5 \end{bmatrix} \end{aligned}$$

where

$$f_\nu(x,y) = \nu_0 + \nu_1 x + \nu_2 x^2 + \nu_3 y + \nu_4 xy + \nu_5 x^2 y + \nu_6 y^2 + \nu_7 xy^2 + \nu_8 x^2 y^2,$$

for $\nu = a, b, c, d$.

The symmetry condition (iii) reduces the number of unknowns by half since $m(x, y)$ becomes

$$m(x,y) = \begin{bmatrix} 1 \\ y \\ y^2 \\ y^3 \\ y^4 \\ y^5 \end{bmatrix}^T \begin{bmatrix} a_0 & b_0 & a_1 & b_1 & a_2 & b_2 \\ b_8 & a_8 & b_7 & a_7 & b_6 & a_6 \\ a_3 & b_3 & a_4 & b_4 & a_5 & b_5 \\ b_5 & a_5 & b_4 & a_4 & b_3 & a_3 \\ a_6 & b_6 & a_7 & b_7 & a_8 & b_8 \\ b_2 & a_2 & b_1 & a_1 & b_0 & a_0 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \\ x^3 \\ x^4 \\ x^5 \end{bmatrix}.$$

For convenience, we denote $\sum_{\nu=a,b} \nu = a + b$. Thus, by (i), we have

$$m(1,1) = 2\sum_{i=0}^{8} \sum_{\nu=a,b} \nu_i = 2\sum_{i=0}^{8} (a_i + b_i) = 1. \tag{1}$$

By (ii), we have the following 13 nonlinear equations

$$\sum_{\nu=a,b} \nu_0\nu_8 = 0 \tag{2}$$

$$\sum_{\nu=a,b} \nu_2\nu_6 = 0 \tag{3}$$

$$\sum_{\nu=a,b} (\nu_1\nu_6 + \nu_2\nu_7) = 0 \tag{4}$$

$$\sum_{\nu=a,b} (\nu_0\nu_7 + \nu_1\nu_8) = 0 \tag{5}$$

$$\sum_{\nu=a,b} (\nu_2\nu_3 + \nu_5\nu_6) = 0 \tag{6}$$

$$\sum_{\nu=a,b} (\nu_0\nu_5 + \nu_3\nu_8) = 0 \tag{7}$$

$$\sum_{\nu=a,b} (\nu_0\nu_6 + \nu_1\nu_7 + \nu_2\nu_8) = 0 \tag{8}$$

$$\sum_{\nu=a,b} (\nu_0\nu_2 + \nu_3\nu_5 + \nu_6\nu_8) = 0 \tag{9}$$

$$\sum_{\nu=a,b} (\nu_1\nu_3 + \nu_2\nu_4 + \nu_4\nu_6 + \nu_5\nu_7) = 0 \tag{10}$$

$$\sum_{\nu=a,b} (\nu_0\nu_4 + \nu_1\nu_5 + \nu_3\nu_7 + \nu_4\nu_8) = 0 \tag{11}$$

$$\sum_{\nu=a,b} (\nu_0\nu_3 + \nu_1\nu_4 + \nu_2\nu_5 + \nu_3\nu_6 + \nu_4\nu_7 + \nu_5\nu_8) = 0 \tag{12}$$

$$\sum_{\nu=a,b} (\nu_0\nu_1 + \nu_1\nu_2 + \nu_3\nu_4 + \nu_4\nu_5 + \nu_6\nu_7 + \nu_7\nu_8) = 0 \tag{13}$$

$$\sum_{i=0}^{8} \sum_{\nu=a,b} \nu_i^2 = \frac{1}{8}. \tag{14}$$

The first moment condition (iv) for $M = 1$ yields the following six linear equations:

$$a_0 + a_1 + a_2 = b_0 + b_1 + b_2 \tag{15}$$
$$a_3 + a_4 + a_5 = b_3 + b_4 + b_5 \tag{16}$$
$$a_6 + a_7 + a_8 = b_6 + b_7 + b_8 \tag{17}$$
$$a_0 + a_3 + a_6 = b_2 + b_5 + b_8 \tag{18}$$
$$a_1 + a_4 + a_7 = b_1 + b_4 + b_7 \tag{19}$$
$$a_2 + a_5 + a_8 = b_0 + b_3 + b_6. \tag{20}$$

We need to find the a_i's and b_i's which satisfy the equations (1)-(20) simultaneously. We proceed by specifying necessary conditions derived from these equations.

LEMMA 2.1. $\displaystyle\sum_{i=0}^{8} a_i = \sum_{i=0}^{8} b_i = \frac{1}{4}.$

Proof. Adding (15)- (17) together, we have $\displaystyle\sum_{i=0}^{8} a_i = \sum_{i=0}^{8} b_i$. By equation (1), the result follows. ■

Next, equations (2), (3), (8), (9), and (14) imply

$$\sum_{\nu=a,b} (\nu_4^2 + (\nu_1 + \nu_7)^2 + (\nu_3 + \nu_5)^2 + (\nu_0 + \nu_2 + \nu_6 + \nu_8)^2) = \frac{1}{8}. \tag{21}$$

We use equations (10) and (11) to get

$$\sum_{\nu=a,b} (\nu_4(\nu_0 + \nu_2 + \nu_6 + \nu_8) + (\nu_1 + \nu_7)(\nu_3 + \nu_5)) = 0. \tag{22}$$

From equations (4), (5), and (13), we have

$$\sum_{\nu=a,b} ((\nu_1 + \nu_7)(\nu_0 + \nu_2 + \nu_6 + \nu_8) + \nu_4(\nu_3 + \nu_5)) = 0. \tag{23}$$

Also, we have, using equations (6), (7), and (12),

$$\sum_{\nu=a,b} (\nu_4(\nu_1 + \nu_7) + (\nu_0 + \nu_2 + \nu_6 + \nu_8)(\nu_3 + \nu_5)) = 0. \tag{24}$$

LEMMA 2.2.

$$\begin{aligned} a_0 + a_2 + a_4 + a_6 + a_8 &= \frac{1}{8} + \frac{1}{4\sqrt{2}} \cos\alpha, \\ b_0 + b_2 + b_4 + b_6 + b_8 &= \frac{1}{8} + \frac{1}{4\sqrt{2}} \sin\alpha. \end{aligned}$$

Proof. By equation (21) and (22), we have

$$\sum_{\nu=a,b}\left((\nu_4+(\nu_0+\nu_2+\nu_6+\nu_8))^2+((\nu_1+\nu_7)+(\nu_3+\nu_5))^2\right)=\frac{1}{8}.$$

By Lemma 2.1 and letting $a^*=a_0+a_2+a_4+a_6+a_8$ and $b^*=b_0+b_2+b_4+b_6+b_8$, we have

$$(a^*)^2+(\frac{1}{4}-a^*)^2+(b^*)^2+(\frac{1}{4}-b^*)^2=\frac{1}{8}$$

or

$$(a^*-\frac{1}{8})^2+(b^*-\frac{1}{8})^2=\frac{1}{32}.$$

Thus, we are able to conclude the proof. ■

Let $\hat{a}=a_1+a_4+a_7$. Observe that equation (19) implies $\hat{a}=b_1+b_4+b_7$. It follows from (21) and (24) that

$$\sum_{\nu=a,b}\left((\nu_4+(\nu_1+\nu_7))^2+((\nu_0+\nu_2+\nu_6+\nu_8)+(\nu_3+\nu_5))^2\right)=\frac{1}{8}.$$

Thus, $2(\hat{a})^2+2(\frac{1}{4}-\hat{a})^2=\frac{1}{8}$. That is, we have $\hat{a}=0$ or $\frac{1}{4}$. Similarly, it follows from (21) and (23) that

$$\sum_{\nu=a,b}\left((\nu_3+\nu_4+\nu_5)^2+((\nu_0+\nu_2+\nu_6+\nu_8)+(\nu_1+\nu_7))^2\right)=\frac{1}{8}.$$

Let $\tilde{a}=a_3+a_4+a_5=b_3+b_4+b_5$ where the second equality comes from equation (16). Thus, $2(\tilde{a})^2+2(\frac{1}{4}-\tilde{a})^2=\frac{1}{8}$. That is, $\tilde{a}=0$ or $\tilde{a}=\frac{1}{4}$. Therefore, we have the following four cases to consider:

- Case 1: $a_1+a_4+a_7=1/4$ and $a_3+a_4+a_5=1/4$
- Case 2: $a_1+a_4+a_7=1/4$ and $a_3+a_4+a_5=0$
- Case 3: $a_1+a_4+a_7=0$ and $a_3+a_4+a_5=1/4$
- Case 4: $a_1+a_4+a_7=0$ and $a_3+a_4+a_5=0$

Since Case 3 is a rotation of Case 2, we will only study Cases 1,2, and 4. As will be discussed, the different cases are actually conditions associated with the zeros of $m(x,y)$ along the axes.

2.1. Complete Solution of Case 1

Let us first consider Case 1 where both $a_1+a_4+a_7=\frac{1}{4}$ and $a_3+a_4+a_5=\frac{1}{4}$. By Lemmas 2.1 and 2.2, we have

$$a_1+a_3+a_5+a_7 \;=\; \frac{1}{8}-\frac{1}{4\sqrt{2}}\cos\alpha,\quad b_1+b_3+b_5+b_7 \;=\; \frac{1}{8}-\frac{1}{4\sqrt{2}}\sin\alpha.$$

However, $(a_1+a_4+a_7)+(a_3+a_4+a_5)=\frac{1}{2}$. It then follows that

$$2a_4=\frac{1}{2}-\frac{1}{8}+\frac{1}{4\sqrt{2}}\cos\alpha \quad\text{or}\quad a_4=\frac{3}{16}+\frac{1}{8\sqrt{2}}\cos\alpha.$$

Similarly, $b_4 = \frac{3}{16} + \frac{1}{8\sqrt{2}} \sin\alpha$. Let us now summarize the above discussion combined with Lemma 2.2 in the following lemma.

LEMMA 2.3.

$$
\begin{aligned}
a_4 &= \frac{3}{16} + \frac{1}{8\sqrt{2}} \cos\alpha, & b_4 &= \frac{3}{16} + \frac{1}{8\sqrt{2}} \sin\alpha,\\
a_1 + a_7 &= \frac{1}{16} - \frac{1}{8\sqrt{2}} \cos\alpha, & b_1 + b_7 &= \frac{1}{16} - \frac{1}{8\sqrt{2}} \sin\alpha,\\
a_3 + a_5 &= \frac{1}{16} - \frac{1}{8\sqrt{2}} \cos\alpha, & b_3 + b_5 &= \frac{1}{16} - \frac{1}{8\sqrt{2}} \sin\alpha,\\
a_0 + a_2 + a_6 + a_8 &= -\frac{1}{16} + \frac{1}{8\sqrt{2}} \cos\alpha, & b_0 + b_2 + b_6 + b_8 &= -\frac{1}{16} + \frac{1}{8\sqrt{2}} \sin\alpha.
\end{aligned}
$$

We are now ready to solve for the 18 unknowns. The complete parameterization of Case 1 is given in Theorem 2.1.

THEOREM 2.1. *For any $\beta, \gamma \in [0, 2\pi]$, let*

$$
\alpha = 2(\beta - \gamma) + \frac{\pi}{4}, \qquad p = \frac{1}{16} - \frac{1}{8\sqrt{2}} \cos\alpha, \quad \text{and} \quad q = \frac{1}{16} - \frac{1}{8\sqrt{2}} \sin\alpha.
$$

If

$$
\begin{aligned}
a_0 &= (-p(1 + \cos(\beta - \gamma)) - q \sin(\beta - \gamma) - \sqrt{p^2 + q^2}(\cos\beta + \cos\gamma))/4\\
a_2 &= (-p(1 - \cos(\beta - \gamma)) + q \sin(\beta - \gamma) - \sqrt{p^2 + q^2}(\cos\beta - \cos\gamma))/4\\
a_6 &= (-p(1 - \cos(\beta - \gamma)) + q \sin(\beta - \gamma) + \sqrt{p^2 + q^2}(\cos\beta - \cos\gamma))/4\\
a_8 &= (-p(1 + \cos(\beta - \gamma)) - q(\sin(\beta - \gamma) + \sqrt{p^2 + q^2}(\cos\beta + \cos\gamma))/4
\end{aligned}
$$

$$
\begin{aligned}
b_0 &= (-q(1 + \cos(\beta - \gamma)) + p \sin(\beta - \gamma) - \sqrt{p^2 + q^2}(\sin\beta + \sin\gamma))/4\\
b_2 &= (-q(1 - \cos(\beta - \gamma)) - p \sin(\beta - \gamma) - \sqrt{p^2 + q^2}(\sin\beta - \sin\gamma))/4\\
b_6 &= (-q(1 - \cos(\beta - \gamma)) - p \sin(\beta - \gamma) + \sqrt{p^2 + q^2}(\sin\beta - \sin\gamma))/4\\
b_8 &= (-q(1 + \cos(\beta - \gamma)) + p \sin(\beta - \gamma) + \sqrt{p^2 + q^2}(\sin\beta + \sin\gamma))/4
\end{aligned}
$$

$$
\begin{aligned}
a_1 &= \frac{p}{2} + \frac{1}{2}\sqrt{p^2 + q^2} \cos\beta, & b_1 &= \frac{q}{2} + \frac{1}{2}\sqrt{p^2 + q^2} \sin\beta\\
a_3 &= \frac{p}{2} + \frac{1}{2}\sqrt{p^2 + q^2} \cos\gamma, & b_3 &= \frac{q}{2} + \frac{1}{2}\sqrt{p^2 + q^2} \sin\gamma\\
a_5 &= \frac{p}{2} - \frac{1}{2}\sqrt{p^2 + q^2} \cos\gamma, & b_3 &= \frac{q}{2} - \frac{1}{2}\sqrt{p^2 + q^2} \sin\gamma\\
a_7 &= \frac{p}{2} - \frac{1}{2}\sqrt{p^2 + q^2} \cos\beta, & b_7 &= \frac{q}{2} - \frac{1}{2}\sqrt{p^2 + q^2} \sin\beta
\end{aligned}
$$

$$
a_4 = \frac{1}{4} - p, \qquad b_4 = \frac{1}{4} - q,
$$

then $m(x, y)$ with these coefficients a_i and b_i satisfies the properties (i), (ii), (iii), and (iv). On the other hand, if $m(x, y)$ satisfies the properties (i), (ii), (iii), (iv), and the conditions of Case 1, then the coefficients a_i and b_i can be expressed in the above format.

Proof. The proof consists of deriving necessary conditions from various combinations of the nonlinear equations which eventually leads to a complete parameterization that satisfies each nonlinear equation individually.

Assume the conditions of Case 1, i.e. $a_1 + a_4 + a_7 = 1/4$ and $a_3 + a_4 + a_5 = 1/4$. Using equations (9) and (14), we have

$$\sum_{\nu=a,b} ((\nu_0 + \nu_2)^2 + (\nu_3 + \nu_5)^2 + (\nu_6 + \nu_8)^2 + \nu_1^2 + \nu_4^2 + \nu_7^2) = \frac{1}{8}.$$

Using (13), the above equation becomes

$$\sum_{\nu=a,b} ((\nu_0 + \nu_1 + \nu_2)^2 + (\nu_3 + \nu_4 + \nu_5)^2 + (\nu_6 + \nu_7 + \nu_8)^2) = \frac{1}{8}.$$

Since $a_3 + a_4 + a_5 = b_3 + b_4 + b_5 = \frac{1}{4}$, we have

$$a_0 + a_1 + a_2 = 0, \ \ a_6 + a_7 + a_8 = 0, \ \ b_0 + b_1 + b_2 = 0, \ \ b_6 + b_7 + b_8 = 0. \quad (25)$$

Similarly, by equations (8) and (14), we have

$$\sum_{\nu=a,b} ((\nu_0 + \nu_6)^2 + (\nu_1 + \nu_7)^2 + (\nu_2 + \nu_8)^2 + \nu_3^2 + \nu_4^2 + \nu_5^2) = \frac{1}{8}.$$

Using (12), the above equation becomes

$$\sum_{\nu=a,b} ((\nu_0 + \nu_3 + \nu_6)^2 + (\nu_1 + \nu_4 + \nu_7)^2 + (\nu_2 + \nu_5 + \nu_8)^2) = \frac{1}{8}.$$

Since $a_1 + a_4 + a_7 = b_1 + b_4 + b_7 = \frac{1}{4}$, we have

$$a_0 + a_3 + a_6 = 0, \ \ a_2 + a_5 + a_8 = 0, \ \ b_0 + b_3 + b_6 = 0, \ \ b_2 + b_5 + b_8 = 0. \quad (26)$$

We now use (25), (26), and the equations in Lemma 2.3 to simplify the equations (2)-(14). By (25), equation (13) becomes

$$\begin{aligned} 0 &= \sum_{\nu=a,b} (-\nu_1^2 + \nu_4(\nu_3 + \nu_5) - \nu_7^2) \\ &= \sum_{\nu=a,b} -(\nu_1 + \nu_7)^2 + \nu_4(\frac{1}{4} - \nu_4) + \nu_1\nu_7 \\ &= \sum_{\nu=a,b} (-(\frac{1}{4} - \nu_4)^2 + \nu_4(\frac{1}{4} - \nu_4) + \nu_1\nu_7) \\ &= \sum_{\nu=a,b} \nu_1\nu_7. \end{aligned}$$

since $\sum_{\nu=a,b}(-(\frac{1}{4} - \nu_4)^2 + \nu_4(\frac{1}{4} - \nu_4)) = 0$ by Lemma 2.3. A similar situation exists for equation (12), thus equations (12) and (13) simplify to

$$a_3a_5 + b_3b_5 = 0, \ \ a_1a_7 + b_1b_7 = 0. \quad (27)$$

We are now able to solve for $a_1, a_3, a_5, a_7, b_1, b_3, b_5,$ and b_7. For simplicity, we denote $a_1 + a_7 = p$ and $b_1 + b_7 = q$. By (27) we have

$$a_1^2 - pa_1 + b_1^2 - b_1 q = 0 \quad \text{or} \quad (a_1 - \frac{p}{2})^2 + (b_1 - \frac{q}{2})^2 = \frac{1}{4}(p^2 + q^2).$$

Thus, we have

$$\begin{aligned} a_1 &= \frac{p}{2} + \frac{1}{2}\sqrt{p^2+q^2}\cos\beta, \quad b_1 = \frac{q}{2} + \frac{1}{2}\sqrt{p^2+q^2}\sin\beta, \\ a_7 &= \frac{p}{2} - \frac{1}{2}\sqrt{p^2+q^2}\cos\beta, \quad b_7 = \frac{q}{2} - \frac{1}{2}\sqrt{p^2+q^2}\sin\beta. \end{aligned}$$

Similarly, we have

$$\begin{aligned} a_3 &= \frac{p}{2} + \frac{1}{2}\sqrt{p^2+q^2}\cos\gamma, \quad b_3 = \frac{q}{2} + \frac{1}{2}\sqrt{p^2+q^2}\sin\gamma, \\ a_5 &= \frac{p}{2} - \frac{1}{2}\sqrt{p^2+q^2}\cos\gamma, \quad b_5 = \frac{q}{2} - \frac{1}{2}\sqrt{p^2+q^2}\sin\gamma. \end{aligned}$$

Next we simplify the equations (2)-(11). Note that the addition of equation (10) and (11) is

$$\sum_{\nu=a,b} (\nu_4(\nu_0 + \nu_2 + \nu_6 + \nu_8) + (\nu_1 + \nu_7)(\nu_3 + \nu_5)) = 0. \tag{28}$$

After substitution of the equations in Lemma 2.3, the left hand side of equation (28) becomes

$$\begin{aligned} & 2(\frac{1}{4} - p)(-p) + 2(-p)(-p) + 2(\frac{1}{4} - q)(-q) + 2(-q)(-q) \\ = \; & 4[p^2 - \frac{1}{8}p + (\frac{1}{16})^2 + q^2 - \frac{1}{8}q + (\frac{1}{16})^2] - 8(\frac{1}{16})^2 \\ = \; & 4[(p - \frac{1}{16})^2 + (q - \frac{1}{16})^2] - \frac{1}{32} \\ = \; & 4[\frac{1}{128}\cos^2\alpha + \frac{1}{128}\sin^2\alpha] - \frac{1}{32} = 0. \end{aligned}$$

That is, the equations (10) and (11) are linearly dependent and consequently only one needs to be considered. We will deal with equation (10) later. Furthermore, for equations (4)-(9), we have the following

$$\begin{aligned} \sum_{\nu=a,b} (\nu_1\nu_6 + \nu_2\nu_7) &= \sum_{\nu=a,b} \nu_1\nu_6 + \nu_1\nu_7 + \nu_2\nu_6 + \nu_2\nu_7 \\ &= \sum_{\nu=a,b} (\nu_1 + \nu_2)(\nu_6 + \nu_7) = \sum_{\nu=a,b} (-\nu_0)(-\nu_8) = \sum_{\nu=a,b} \nu_0\nu_8, \end{aligned}$$

$$\begin{aligned} \sum_{\nu=a,b} (\nu_0\nu_7 + \nu_1\nu_8) &= \sum_{\nu=a,b} \nu_0\nu_7 + \nu_0\nu_8 + \nu_1\nu_7 + \nu_1\nu_8 \\ &= \sum_{\nu=a,b} (\nu_0 + \nu_1)(\nu_7 + \nu_8) = \sum_{\nu=a,b} (-\nu_2)(-\nu_6) = \sum_{\nu=a,b} \nu_2\nu_6, \end{aligned}$$

$$\sum_{\nu=a,b}(\nu_2\nu_3+\nu_5\nu_6)=\sum_{\nu=a,b}\nu_2\nu_3+\nu_2\nu_6+\nu_3\nu_5+\nu_5\nu_6$$
$$=\sum_{\nu=a,b}(\nu_2+\nu_5)(\nu_3+\nu_6)=\sum_{\nu=a,b}(-\nu_8)(-\nu_0)=\sum_{\nu=a,b}\nu_0\nu_8,$$

$$\sum_{\nu=a,b}(\nu_0\nu_5+\nu_3\nu_8)=\sum_{\nu=a,b}\nu_0\nu_5+\nu_0\nu_8+\nu_3\nu_5+\nu_3\nu_8$$
$$=\sum_{\nu=a,b}(\nu_0+\nu_3)(\nu_5+\nu_8)=\sum_{\nu=a,b}(-\nu_6)(-\nu_2)=\sum_{\nu=a,b}\nu_2\nu_6,$$

$$\sum_{\nu=a,b}(\nu_0\nu_6+\nu_1\nu_7+\nu_2\nu_8)=\sum_{\nu=a,b}\nu_0\nu_6+\nu_2\nu_8=\sum_{\nu=a,b}\nu_0\nu_6+\nu_0\nu_8+\nu_2\nu_6+\nu_2\nu_8$$
$$=\sum_{\nu=a,b}(\nu_0+\nu_2)(\nu_6+\nu_8)=\sum_{\nu=a,b}(-\nu_1)(-\nu_7)=\sum_{\nu=a,b}\nu_1\nu_7,$$

$$\sum_{\nu=a,b}(\nu_0\nu_2+\nu_3\nu_5+\nu_6\nu_8)=\sum_{\nu=a,b}\nu_0\nu_2+\nu_0\nu_8+\nu_2\nu_6+\nu_6\nu_8$$
$$=\sum_{\nu=a,b}(\nu_0+\nu_6)(\nu_2+\nu_8)=\sum_{\nu=a,b}(-\nu_3)(-\nu_5)=\sum_{\nu=a,b}\nu_3\nu_5.$$

Therefore, only $\sum_{\nu=a,b}\nu_0\nu_8=0$ and $\sum_{\nu=a,b}\nu_2\nu_6=0$ remain to be solved in addition to (10) since equation (14) follows from (1)-(13).

In order to solve equations (2) and (3), note that (25) and (26) imply $a_0=a_5+a_8-a_1$ and $b_0=b_5+b_8-b_1$. Thus, (2) becomes

$$a_8^2+a_8(a_5-a_1)+b_8^2+b_8(b_5-b_1)=0 \tag{29}$$

while (3) becomes

$$(a_5+a_8)(a_7+a_8)+(b_5+b_8)(b_7+b_8)=0$$

or

$$a_8^2+a_8(a_5+a_7)+b_8^2+b_8(b_5+b_7)=-a_5a_7-b_5b_7. \tag{30}$$

The subtraction of (29) from (30) yields

$$a_8(a_1+a_7)+b_8(b_1+b_7)=-a_5a_7-b_5b_7.$$

So, we have

$$pa_8+qb_8=-a_5a_7-b_5b_7. \tag{31}$$

The addition of (29) and (30) yields

$$a_8^2+(a_5+\frac{a_7-a_1}{2})a_8+b_8^2+(b_5+\frac{b_7-b_1}{2})b_8=-\frac{1}{2}(a_5a_7+b_5b_7)$$

which is

$$(a_8 + \frac{1}{2}(a_5 + \frac{a_7 - a_1}{2}))^2 + (b_8 + \frac{1}{2}(b_5 + \frac{b_7 - b_1}{2}))^2 = R_1 \tag{32}$$

for some known value R_1. In addition, equation (31) can be rewritten as

$$p(a_8 + \frac{1}{2}(a_5 + \frac{a_7 - a_1}{2})) + q(b_8 + \frac{1}{2}(b_5 + \frac{b_7 - b_1}{2})) = R_2 \tag{33}$$

for some known value R_2. Equations (32) and (33) can be solved simultaneously. Thus, we obtain the expression for the a_i's and b_i's given in Theorem 2.1.

Finally, to satisfy (10), we put these a_i and b_i into (10) and simplify the equation yielding

$$\cos(\beta - \gamma) = \frac{1}{\sqrt{2}} \cos(\alpha - \beta + \gamma) + \frac{1}{\sqrt{2}} \sin(\alpha - \beta + \gamma).$$

Solving, we find that $\alpha = 2(\beta - \gamma) + \pi/4$.

The above discussion shows that if $m(x, y)$ satisfies the properties (i), (ii), (iii), (iv), and the conditions of Case 1, then its coefficients a_i and b_i can be expressed as the two-parameter family given in the statement of Theorem 2.1.

On the other hand, to verify that $m(x, y)$ with the coefficients a_i and b_i satisfies the properties (i), (ii), (iii), (iv), and the conditions of Case 1, we just substitute the solutions back into equations (1)-(20). This completes the proof. ■

2.2. Solution of Case 2

In this section, we consider Case 2 where $a_1 + a_4 + a_7 = 1/4$ and $a_3 + a_4 + a_5 = 0$. By Lemmas 2.1 and 2.2, we have $a_1 + a_3 + a_5 + a_7 = \frac{1}{8} - \frac{1}{4\sqrt{2}} \cos \alpha$. So, we have

$$a_4 = \frac{1}{16} + \frac{1}{8\sqrt{2}} \cos \alpha \text{ and } b_4 = \frac{1}{16} + \frac{1}{8\sqrt{2}} \sin \alpha.$$

By (9),(13), and (14), we have

$$\sum_{\nu = a,b} \left((\nu_0 + \nu_1 + \nu_2)^2 + (\nu_3 + \nu_4 + \nu_5)^2 + (\nu_6 + \nu_7 + \nu_8)^2 \right) = \frac{1}{8}.$$

The assumptions of Case 2 imply

$$\sum_{\nu = a,b} \left((\nu_0 + \nu_1 + \nu_2)^2 + (\nu_6 + \nu_7 + \nu_8)^2 \right) = \frac{1}{8},$$

i.e.

$$(a_0 + a_1 + a_2)^2 + (a_6 + a_7 + a_8)^2 + (b_0 + b_1 + b_2)^2 + (b_6 + b_7 + b_8)^2 = \frac{1}{8}.$$

By (15) and (17), we have

$$(a_0 + a_1 + a_2)^2 + (a_6 + a_7 + a_8)^2 = \frac{1}{16}.$$

By Lemma 2.1, the above equation becomes

$$(\frac{1}{4}-(a_6+a_7+a_8))^2+(a_6+a_7+a_8)^2=\frac{1}{16}.$$

It follows that $a_6+a_7+a_8=0$ or $a_6+a_7+a_8=\frac{1}{4}$. Thus, Case 2 branches out into two subcases.

- Subcase 2a: $a_1+a_4+a_7=\dfrac{1}{4},\ a_3+a_4+a_5=0,\ a_6+a_7+a_8=0,\ a_0+a_1+a_2=\dfrac{1}{4}.$
- Subcase 2b: $a_1+a_4+a_7=\dfrac{1}{4},\ a_3+a_4+a_5=0,\ a_6+a_7+a_8=\dfrac{1}{4},\ a_0+a_1+a_2=0.$

We only consider subcase $2a$ here. The subcase $2b$ can be treated similarly and is left to the interested reader. Theorem 2.2 gives the complete solution of Subcase 2a.

THEOREM 2.2.
1) For any $\gamma\in[0,2\pi]$, $a_3=b_3=a_4=b_4=a_5=b_5=0$,

$$\begin{aligned}
a_1 &= \frac{3}{16}-\frac{1}{8\sqrt{2}}\cos\gamma, \quad a_7=\frac{1}{16}+\frac{1}{8\sqrt{2}}\cos\gamma,\\
a_0 &= \frac{1}{32}\left(1+\sqrt{2}\cos\gamma\pm\sqrt{2+\sqrt{2}(\cos\gamma+\sin\gamma)}\right),\\
a_2 &= \frac{1}{32}\left(1+\sqrt{2}\cos\gamma\mp\sqrt{2+\sqrt{2}(\cos\gamma+\sin\gamma)}\right),\\
a_6 &= \frac{1}{32}\left(-1-\sqrt{2}\cos\gamma\mp\sqrt{2+\sqrt{2}(\cos\gamma+\sin\gamma)}\right),\\
a_8 &= \frac{1}{32}\left(-1-\sqrt{2}\cos\gamma\pm\sqrt{2+\sqrt{2}(\cos\gamma+\sin\gamma)}\right),\\
b_1 &= \frac{3}{16}-\frac{1}{8\sqrt{2}}\sin\gamma, \quad b_7=\frac{1}{16}+\frac{1}{8\sqrt{2}}\sin\gamma,\\
b_0 &= \frac{1}{32}\left(1+\sqrt{2}\sin\gamma\mp\sqrt{2+\sqrt{2}(\cos\gamma+\sin\gamma)}\right),\\
b_2 &= \frac{1}{32}\left(1+\sqrt{2}\sin\gamma\pm\sqrt{2+\sqrt{2}(\cos\gamma+\sin\gamma)}\right),\\
b_6 &= \frac{1}{32}\left(-1-\sqrt{2}\sin\gamma\pm\sqrt{2+\sqrt{2}(\cos\gamma+\sin\gamma)}\right),\\
b_8 &= \frac{1}{32}\left(-1-\sqrt{2}\sin\gamma\mp\sqrt{2+\sqrt{2}(\cos\gamma+\sin\gamma)}\right).
\end{aligned}$$

2) For any $\alpha\in[0,\frac{\pi}{4})\cup(\frac{\pi}{4},2\pi]$, *let*

$$\begin{aligned}
\gamma &= \begin{cases} -\frac{1}{2}\alpha+\frac{7\pi}{8} \ \text{ or } \ \frac{1}{2}\alpha-\frac{3\pi}{8} & \text{if } \ 0\le\alpha<\frac{\pi}{4} \\ \frac{1}{2}\alpha+\frac{5\pi}{8} \ \text{ or } \ -\frac{1}{2}\alpha-\frac{\pi}{8} & \text{if } \ \frac{\pi}{4}<\alpha\le 2\pi \end{cases}\\
p &= -\frac{1}{16}-\frac{1}{8\sqrt{2}}\cos\alpha, \quad q=-\frac{1}{16}-\frac{1}{8\sqrt{2}}\sin\alpha,\\
s &= \sqrt{32(p^2+q^2)}, \quad t=\sqrt{(p+\frac{1}{8})^2+(q+\frac{1}{8})^2},
\end{aligned}$$

$$a_0 = -\frac{1}{32}\left(-1+8p+s+\frac{t}{p-q}\left[(-s-8p+8q)\cos\gamma + s\sin\gamma\right]\right)$$
$$a_2 = \frac{1}{32}\left(1-8p+s+\frac{t}{p-q}\left[(-s+8p-8q)\cos\gamma + s\sin\gamma\right]\right)$$
$$a_6 = -\frac{1}{32}\left(1+8p+s+\frac{t}{p-q}\left[(s+8p-8q)\cos\gamma - s\sin\gamma\right]\right)$$
$$a_8 = \frac{1}{32}\left(-1-8p+s+\frac{t}{p-q}\left[(s-8p+8q)\cos\gamma - s\sin\gamma\right]\right)$$
$$b_0 = \frac{1}{32}\left(1-8q+s+\frac{t}{p-q}\left[(s+8p-8q)\sin\gamma - s\cos\gamma\right]\right)$$
$$b_2 = -\frac{1}{32}\left(-1+8q+s+\frac{t}{p-q}\left[(s-8p+8q)\sin\gamma - s\cos\gamma\right]\right)$$
$$b_6 = \frac{1}{32}\left(-1-8q+s+\frac{t}{p-q}\left[(-s-8p+8q)\sin\gamma + s\cos\gamma\right]\right)$$
$$b_8 = -\frac{1}{32}\left(1+8q+s+\frac{t}{p-q}\left[(-s+8p-8q)\sin\gamma + s\cos\gamma\right]\right)$$

$$\begin{aligned}
a_1 &= \frac{1}{16}(3+8p-8t\cos\gamma), & b_1 &= \frac{1}{16}(3+8q-8t\sin\gamma)\\
a_7 &= \frac{1}{16}(1+8p+8t\cos\gamma), & b_7 &= \frac{1}{16}(1+8q+8t\sin\gamma)\\
a_3 &= \frac{1}{16}(8p+s), & b_3 &= \frac{1}{16}(8q-s),\\
a_5 &= \frac{1}{16}(8p-s), & b_5 &= \frac{1}{16}(8q+s)\\
a_4 &= -p, & b_4 &= -q.
\end{aligned}$$

3) For $\alpha = \frac{\pi}{4}$,

$$(c_{jk})_{j,k} = \frac{1}{8}\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & -1 & 1 & 1 & -1 & 0\\ 0 & -1 & 1 & 1 & -1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix} \quad \text{or} \quad \frac{1}{16}\begin{bmatrix} 1 & 1 & 2 & 2 & 1 & 1\\ -1 & 1 & 0 & 0 & 1 & -1\\ 0 & -2 & 2 & 2 & -2 & 0\\ 0 & -2 & 2 & 2 & -2 & 0\\ -1 & 1 & 0 & 0 & 1 & -1\\ 1 & 1 & 2 & 2 & 1 & 1 \end{bmatrix}.$$

If $m(x,y)$ has the coefficients given by 1), 2), or 3), then $m(x,y)$ satisfies the properties of (i), (ii), (iii), and (iv). Conversely, if $m(x,y)$ satisfies the properties of (i), (ii), (iii), (iv), and the conditions of Subcase 2a, then the coefficients of $m(x,y)$ can be expressed as 1), 2), or 3).

Proof. Assume the conditions of Subcase 2a, i.e.

$$a_1+a_4+a_7 = \frac{1}{4}, \quad a_3+a_4+a_5 = 0, \quad a_6+a_7+a_8 = 0, \quad a_0+a_1+a_2 = \frac{1}{4}.$$

We immediately have from Lemma 2.2

$$a_4 = \frac{1}{16} + \frac{1}{8\sqrt{2}} \cos \alpha, \quad b_4 = \frac{1}{16} + \frac{1}{8\sqrt{2}} \sin \alpha.$$

By equations (8), (12) and (14), we have

$$\sum_{\nu=a,b} (\nu_0 + \nu_3 + \nu_6)^2 + (\nu_1 + \nu_4 + \nu_7)^2 + (\nu_2 + \nu_5 + \nu_8)^2 = \frac{1}{8}.$$

It follows that

$$\begin{aligned} a_0 + a_3 + a_6 &= 0, \quad b_0 + b_3 + b_6 = 0, \\ a_2 + a_5 + a_8 &= 0, \quad b_2 + b_5 + b_8 = 0. \end{aligned} \tag{34}$$

We now simplify the 13 nonlinear equations (2) -(14). With (2), (3), and (34), we may simplify (9) as follows:

$$\begin{aligned} 0 &= \sum_{\nu=a,b} (\nu_0\nu_2 + \nu_3\nu_5 + \nu_6\nu_8) = \sum_{\nu=a,b} (\nu_0 + \nu_6)(\nu_2 + \nu_8) + \nu_3\nu_5 \\ &= \sum_{\nu=a,b} (-\nu_3)(-\nu_5) + \nu_3\nu_5 = 2\sum_{\nu=a,b} \nu_3\nu_5. \end{aligned} \tag{35}$$

Recall $a_3 + a_4 + a_5 = 0$ and $b_3 + b_4 + b_5 = 0$. We can now solve for a_3, a_5, b_3, b_5. Indeed, (35) can be rewritten as $(a_4 + a_5)a_5 + (b_4 + b_5)b_5 = 0$. After simplifying, we have

$$(a_5 + \frac{a_4}{2})^2 + (b_5 + \frac{b_4}{2})^2 = \frac{1}{4}(a_4^2 + b_4^2).$$

Thus, we have

$$\begin{aligned} a_3 &= \frac{p}{2} + \frac{1}{2}\sqrt{p^2+q^2}\cos\beta, \quad b_3 = \frac{q}{2} + \frac{1}{2}\sqrt{p^2+q^2}\sin\beta \\ a_5 &= \frac{p}{2} - \frac{1}{2}\sqrt{p^2+q^2}\cos\beta, \quad b_5 = \frac{q}{2} - \frac{1}{2}\sqrt{p^2+q^2}\sin\beta, \end{aligned}$$

where $p = -a_4$ and $q = -b_4$ (different p and q from section 2.1). We note that

$$p^2 + q^2 + \frac{1}{8}(p+q) = 0. \tag{36}$$

Similarly, we may simplify (8) to be

$$\begin{aligned} 0 &= \sum_{\nu=a,b} (\nu_1\nu_7 + \nu_0\nu_6 + \nu_2\nu_8) = \sum_{\nu=a,b} \nu_1\nu_7 + (\nu_0 + \nu_2)(\nu_6 + \nu_8) \\ &= \sum_{\nu=a,b} \nu_1\nu_7 + (\frac{1}{4} - \nu_1)(-\nu_7) = 2\sum_{\nu=a,b} (\nu_1\nu_7 - \frac{1}{8}\nu_7). \end{aligned} \tag{37}$$

Recall $a_1 + a_7 = \frac{1}{4} - a_4$ and $b_1 + b_7 = \frac{1}{4} - b_4$. We may solve for a_1 and b_1 to get $a_1 = \frac{1}{4} - a_4 - a_7$ and $b_1 = \frac{1}{4} - b_4 - b_7$. Putting them into (37), we have

$$\left(\frac{1}{4} - a_4 - a_7\right) a_7 - \frac{a_7}{8} + \left(\frac{1}{4} - b_4 - b_7\right) b_7 - \frac{b_7}{8} = 0$$

or

$$\left(a_7 - \frac{\frac{1}{8} - a_4}{2}\right)^2 + \left(b_7 - \frac{\frac{1}{8} - b_4}{2}\right)^2 = \frac{\left(a_4 - \frac{1}{8}\right)^2 + \left(b_4 - \frac{1}{8}\right)^2}{4}.$$

It follows that

$$\begin{aligned}
a_7 &= \frac{p}{2} + \frac{1}{16} + \frac{1}{2}\sqrt{\left(p + \frac{1}{8}\right)^2 + \left(q + \frac{1}{8}\right)^2}\ \cos\gamma, \\
b_7 &= \frac{q}{2} + \frac{1}{16} + \frac{1}{2}\sqrt{\left(p + \frac{1}{8}\right)^2 + \left(q + \frac{1}{8}\right)^2}\ \sin\gamma, \\
a_1 &= \frac{p}{2} + \frac{3}{16} - \frac{1}{2}\sqrt{\left(p + \frac{1}{8}\right)^2 + \left(q + \frac{1}{8}\right)^2}\ \cos\gamma, \\
b_1 &= \frac{q}{2} + \frac{3}{16} - \frac{1}{2}\sqrt{\left(p + \frac{1}{8}\right)^2 + \left(q + \frac{1}{8}\right)^2}\ \sin\gamma.
\end{aligned}$$

We add the left-hand side of (10) and (11) together to get

$$\begin{aligned}
&\sum_{\nu=a,b} \left(\nu_0 + \nu_2\right)\nu_4 + \nu_1\left(\nu_3 + \nu_5\right) + \nu_4\left(\nu_6 + \nu_8\right) + \nu_7\left(\nu_3 + \nu_5\right) \\
&\quad = \sum_{\nu=a,b} \nu_4\left(\frac{1}{4} - \nu_1\right) + \nu_1\left(-\nu_4\right) + \nu_4\left(-\nu_7\right) + \nu_7\left(-\nu_4\right) \\
&\quad = \sum_{\nu=a,b} \nu_4\left[\frac{1}{4} - 2\left(\nu_1 + \nu_7\right)\right] = \sum_{\nu=a,b} \nu_4\left(2\nu_4 - \frac{1}{4}\right) \\
&\quad = 2\left(2a_4^2 - \frac{1}{4}a_4\right) + 2\left(2b_4^2 - \frac{1}{8}b_4\right) \\
&\quad = 4\left[p^2 + q^2 + \frac{1}{8}\left(p + q\right)\right] = 0.
\end{aligned}$$

That is, only one of (10) and (11) needs to be solved. Thus, we deal with (10) later.

Turning our attention to equation (13). We have by (35) and (36)

$$\begin{aligned}
&\sum_{\nu=a,b} \nu_0\nu_1 + \nu_1\nu_2 + \nu_3\nu_4 + \nu_4\nu_5 + \nu_6\nu_7 + \nu_7\nu_8 \\
&\quad = \sum_{\nu=a,b} \left(\nu_0 + \nu_2\right)\nu_1 + \left(\nu_3 + \nu_5\right)\nu_4 + \nu_7\left(\nu_6 + \nu_8\right) \\
&\quad = \sum_{\nu=a,b} \left(\frac{1}{4} - \nu_1\right)\nu_1 - \nu_4^2 - \nu_7^2 \\
&\quad = \sum_{\nu=a,b} -2\nu_1\nu_7 - \nu_1^2 - \nu_7^2 - \nu_4^2 + \frac{1}{4}(\nu_1 + \nu_7) \\
&\quad = \sum_{\nu=a,b} \left(-\left(\nu_1 + \nu_7\right)^2 - \nu_4^2 + \frac{1}{4}(\nu_1 + \nu_7)\right)
\end{aligned}$$

$$= -\sum_{\nu=a,b}\left(\frac{1}{4}-\nu_4\right)^2+\nu_4^2-\frac{1}{4}\left(\frac{1}{4}-\nu_4\right) = 0.$$

That is, (13) holds for these a_4, a_1, a_7, b_1, b_7. Similarly, (12) is satisfied in that

$$\begin{aligned}
&\sum_{\nu=a,b}\left(\nu_0\nu_3+\nu_1\nu_4+\nu_2\nu_5+\nu_3\nu_6+\nu_4\nu_7+\nu_5\nu_8\right)\\
&= \sum_{\nu=a,b}-\nu_3\left(\nu_0+\nu_6\right)+\nu_4\left(\nu_1+\nu_7\right)+\nu_5\left(\nu_2+\nu_8\right)\\
&= \sum_{\nu=a,b}\left(-\nu_3^2+\nu_4\left(\frac{1}{4}-\nu_4\right)-\nu_5^2\right)\\
&= \sum_{\nu=a,b}-\left(\nu_3+\nu_5\right)^2-a_4^2+\frac{\nu_4}{4}\\
&= -\sum_{\nu=a,b}\left(2\nu_4^2-\frac{\nu_4}{4}\right)=0.
\end{aligned}$$

We now show that equations (3), (9), (34) and (35) imply (6). The addition of (6) and (9) yields

$$\sum_{\nu=a,b}\nu_6\left(\nu_5+\nu_8\right)+\nu_2\left(\nu_0+\nu_3\right)+\nu_3\nu_5=\sum_{\nu=a,b}-2\nu_2\nu_6+\nu_3\nu_5=0.$$

Similarly, the equations (2), (9) and (35) imply (7). Since (8) holds for these a_1, b_7, a_7, b_7 under the assumptions of (2) and (3), we further simplify (4) by adding (4) and (8) together. That is,

$$\begin{aligned}
0 &= \sum_{\nu=a,b}\nu_2\nu_7+\nu_1\nu_6+\nu_1\nu_7+\nu_0\nu_6+\nu_2\nu_8\\
&= \sum_{\nu=a,b}\nu_6\left(\nu_0+\nu_1\right)+\nu_2\left(\nu_7+\nu_8\right)+\nu_1\nu_7\\
&= \sum_{\nu=a,b}\nu_6\left(\frac{1}{4}-\nu_2\right)+\nu_2\left(-\nu_6\right)+\nu_1\nu_7\\
&= \sum_{\nu=a,b}\left(\frac{1}{4}\nu_6+\nu_1\nu_7\right)=\sum_{\nu=a,b}\left(\frac{1}{4}\nu_6+\frac{1}{8}\nu_7\right).
\end{aligned}$$

Similarly, the sum of equations (5) and (8) is equivalent to

$$\sum_{\nu=a,b}\left(\nu_1\nu_7+\frac{1}{4}\nu_8\right)=\sum_{\nu=a,b}\left(\frac{1}{8}\nu_7+\frac{1}{4}\nu_8\right)=0. \tag{38}$$

However, equations (37) and (38) are equivalent by using $a_6+a_7+a_8=0$ and $b_6+b_7+b_8=0$. Thus, we only need to consider one of them.

In summary, we only need to solve the following equations

$$a_0a_8+b_0b_8=0, \quad a_2a_6+b_2b_8=0,$$

$$\sum_{\nu=a,b}(\nu_7+2\nu_8)=0,\quad \sum_{\nu=a,b}(\nu_0\nu_4+\nu_4\nu_8+\nu_1\nu_5+\nu_3\nu_7)=0.$$

Using the linear relationships, we have

$$\begin{aligned} a_0 &= \frac{1}{4}-a_1+a_5+a_8,\;\; b_0=\frac{1}{4}-b_1+b_5+b_8 \\ a_2 &= -a_5-a_8,\quad b_2=-b_5-b_8,\quad a_6=-a_7-a_8,\quad b_6=-b_7-b_8. \end{aligned}$$

Putting these linear equations in (2) and (3), we get

$$\begin{aligned} a_8^2+(a_5-a_1+\frac{1}{4})a_8+b_8^2+(b_5-b_1+\frac{1}{4})b_8 &= 0 \\ a_8^2+(a_5+a_7)a_8+a_5a_7+b_8^2+(b_5+b_7)b_8+b_5b_7 &= 0. \end{aligned}$$

Subtracting the first one of the above two equations from the second one, and using $a_1+a_7=p+\frac{1}{4}$, $b_1+b_7=q+\frac{1}{4}$, we get

$$pa_8+qb_8=-a_5a_7-b_5b_7.$$

Using these linear relationships and (11), we get

$$pa_8+qb_8=\frac{1}{2}\sum_{\nu=a,b}(\nu_1\nu_5+\nu_3\nu_7(\nu_5-\nu_1+\frac{1}{4})\nu_4).$$

It follows that

$$\sum_{\nu=a,b}(2\nu_5\nu_7+\nu_1\nu_5+\nu_3\nu_7+(\nu_5-\nu_1+\frac{1}{4})\nu_4)=0$$

which can be simplified to be

$$\sum_{\nu=a,b}(\frac{1}{4}\nu_5+\nu_4^2)=0$$

That is, we have

$$\begin{aligned} 0 &= a_4^2+b_4^2+\frac{1}{4}(a_5+b_5) \\ &= \frac{1}{8}(a_4+b_4)+\frac{1}{4}(p+q+\frac{1}{2}\sqrt{p^2+q^2}(\cos\beta+\sin\beta) \\ &= \frac{1}{\sqrt{2}}\sqrt{p^2+q^2}\sin(\frac{\pi}{4}+\beta). \end{aligned}$$

It follows that either $\beta=-\frac{\pi}{4}$, $\frac{3\pi}{4}$ or $p^2+q^2=0$ which the latter occurs when $\alpha=-\frac{3\pi}{4}$.

We first consider $\alpha=-\frac{3\pi}{4}$. In this situation, we have $p=0$ and $q=0$. It follows that $a_3=b_3=a_4=b_4=a_5=b_5=0$. It is clear that $pa_8+qb_8=-\sum_{\nu=a,b}\nu_5\nu_7$ holds. Thus, we only have two equations to solve

$$a_0a_8+b_0b_8=0,\quad a_8+b_8=-\frac{1}{2}(a_7+b_7).$$

Using $a_0 = \frac{1}{4} - a_1 + a_8$ and $b_0 = \frac{1}{4} - b_1 + b_8$, we solve the above equations and get

$$\begin{aligned}
a_8 &= \frac{1}{32}\Big(-1 - \sqrt{2}\cos\gamma \pm \sqrt{2+\sqrt{2}(\cos\gamma+\sin\gamma)}\Big) \\
b_8 &= \frac{1}{32}\Big(-1 - \sqrt{2}\sin\gamma \mp \sqrt{2+\sqrt{2}(\cos\gamma+\sin\gamma)}\Big) \\
a_0 &= \frac{1}{32}\Big(1 + \sqrt{2}\cos\gamma \pm \sqrt{2+\sqrt{2}(\cos\gamma+\sin\gamma)}\Big) \\
b_0 &= \frac{1}{32}\Big(1 + \sqrt{2}\sin\gamma \mp \sqrt{2+\sqrt{2}(\cos\gamma+\sin\gamma)}\Big).
\end{aligned}$$

Using the linear relationships, we have

$$\begin{aligned}
a_2 &= \frac{1}{32}\Big(1 + \sqrt{2}\cos\gamma \mp \sqrt{2+\sqrt{2}(\cos\gamma+\sin\gamma)}\Big) \\
b_2 &= \frac{1}{32}\Big(1 + \sqrt{2}\sin\gamma \pm \sqrt{2+\sqrt{2}(\cos\gamma+\sin\gamma)}\Big) \\
a_6 &= \frac{1}{32}\Big(-1 - \sqrt{2}\cos\gamma \mp \sqrt{2+\sqrt{2}(\cos\gamma+\sin\gamma)}\Big) \\
b_6 &= \frac{1}{32}\Big(-1 - \sqrt{2}\sin\gamma \pm \sqrt{2+\sqrt{2}(\cos\gamma+\sin\gamma)}\Big) \\
a_1 &= \frac{3}{16} - \frac{\cos\gamma}{8\sqrt{2}},\ \ b_1 = \frac{3}{16} - \frac{\sin\gamma}{8\sqrt{2}},\ \ a_7 = \frac{1}{16} + \frac{\cos\gamma}{8\sqrt{2}},\ \ b_7 = \frac{1}{16} + \frac{\sin\gamma}{8\sqrt{2}}.
\end{aligned}$$

Next we consider $\beta = -\frac{\pi}{4}$ or $\beta = \frac{3\pi}{4}$. We only need to consider $\beta = -\frac{\pi}{4}$ while $p^2+q^2 \neq 0$ because the other case is a rotation of this one. For $\beta = -\frac{\pi}{4}$, we have three equations to solve:

$$a_0a_8 + b_0b_8 = 0, \quad pa_8 + qb_8 = -\sum_{\nu=a,b} a_5a_7, \quad a_8 + b_8 = -\frac{1}{2}(a_7+b_7). \tag{39}$$

Assuming $p \neq q$, i.e. $\alpha \neq \frac{\pi}{4}$, we can solve the second and third ones in the above three equations for a_8 and b_8: The solutions for a_8 and b_8 are

$$\begin{aligned}
b_8 &= \frac{1}{q-p}\Big(-\frac{p}{2}(a_7+b_7) + \sum_{\nu=a,b} a_5a_7\Big) \\
a_8 &= -\frac{1}{2}(a_7+b_7) + \frac{1}{p-q}\Big(-\frac{p}{2}(a_7+b_7) + \sum_{\nu=a,b} a_5a_7\Big)
\end{aligned}$$

leaving the relationship between α and γ as $a_0a_8 + b_0b_8 = 0$. Upon substitution and simplification, we have

$$\sin(\alpha+\frac{\pi}{4}) - 2\sin(\gamma+\frac{\pi}{4})\sqrt{2(1-\sin(\alpha+\frac{\pi}{4})} + \sin(2\gamma) - 2 = 0. \tag{40}$$

This equation (40) has a one parameter family of solutions given by

$$\gamma = \begin{cases} -\frac{1}{2}\alpha + \frac{7\pi}{8} \text{ or } \frac{1}{2}\alpha - \frac{3\pi}{8} & \text{if } 0 < \alpha < \frac{\pi}{4} \\ \frac{1}{2}\alpha + \frac{5\pi}{8} \text{ or } -\frac{1}{2}\alpha - \frac{\pi}{8} & \text{if } \frac{\pi}{4} < \alpha < 2\pi. \end{cases}$$

The final case is when $\beta = -\frac{\pi}{4}$ and $p = q$ (i.e. $\alpha = \frac{\pi}{4}$) which implies that

$$a_1 = b_1 = a_4 = b_4 = \frac{1}{8}, a_3 = a_7 = b_5 = b_7 = 0, a_5 = b_3 = -\frac{1}{8}.$$

This reduces (39) to

$$a_0 a_8 + b_0 b_8 = 0, \quad a_8 + b_8 = 0,$$

which has two solutions: $a_8 = b_8 = 0$ or $a_8 = \frac{1}{16}$ and $b_8 = -\frac{1}{16}$ yielding two rational solutions:

$$\frac{1}{8}\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & -1 & 0 \\ 0 & -1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}, \quad \frac{1}{16}\begin{bmatrix} 1 & 1 & 2 & 2 & 1 & 1 \\ -1 & 1 & 0 & 0 & 1 & -1 \\ 0 & -2 & 2 & 2 & -2 & 0 \\ 0 & -2 & 2 & 2 & -2 & 0 \\ -1 & 1 & 0 & 0 & 1 & -1 \\ 1 & 1 & 2 & 2 & 1 & 1 \end{bmatrix}.$$

■

2.3. Solution of Case 4

Finally, we consider Case 4 where both $a_1 + a_4 + a_7 = 0$ and $a_3 + a_4 + a_5 = 0$. From equations (16) and (19) we have $b_3 + b_4 + b_5 = 0$ and $b_1 + b_4 + b_7 = 0$. As before, using equations (9), (13) and (14), we have

$$\sum_{\nu=a,b} ((\nu_0 + \nu_1 + \nu_2)^2 + (\nu_3 + \nu_4 + \nu_5)^2 + (\nu_6 + \nu_7 + \nu_8)^2) = \frac{1}{8}.$$

It follows that $\sum_{\nu=a,b}((\nu_0 + \nu_1 + \nu_2)^2 + (\nu_6 + \nu_7 + \nu_8)^2) = \frac{1}{8}$. Using equations (15) and (17), we have

$$(a_0 + a_1 + a_2)^2 + (a_6 + a_7 + a_8)^2 = \frac{1}{16}.$$

By Lemma 2.1, the above equation is

$$(\frac{1}{4} - (a_6 + a_7 + a_8))^2 + (a_6 + a_7 + a_8)^2 = \frac{1}{16}.$$

Thus, $a_6 + a_7 + a_8 = \frac{1}{4}$ or $a_6 + a_7 + a_8 = 0$.

Also, using equations (8), (12) and (14), we have

$$\sum_{\nu=a,b} \left((\nu_0 + \nu_3 + \nu_6)^2 + (\nu_1 + \nu_4 + \nu_7)^2 + (\nu_2 + \nu_5 + \nu_8)^2\right) = \frac{1}{8}.$$

It follows that $\sum_{\nu=a,b} \left((\nu_0 + \nu_3 + \nu_6)^2 + (\nu_2 + \nu_5 + \nu_8)^2\right) = \frac{1}{8}$. So, we have

$$(a_0 + a_3 + a_6)^2 + (a_2 + a_5 + a_8)^2 = \frac{1}{16}.$$

Again by Lemma 2.1, $\left(\frac{1}{4} - (a_0 + a_3 + a_6)\right)^2 + (a_0 + a_3 + a_6)^2 = \frac{1}{16}$. Thus, $a_0 + a_3 + a_6 = 0$ or $a_0 + a_3 + a_6 = \frac{1}{4}$.

Therefore, we have four subcases to consider. In addition to

$$a_1 + a_4 + a_7 = 0, a_3 + a_4 + a_5 = 0, b_1 + b_4 + b_7 = 0, b_3 + b_4 + b_5 = 0,$$

we have

- Subcase 4a:

$$a_0 + a_1 + a_2 = \frac{1}{4}, \quad a_0 + a_3 + a_6 = \frac{1}{4}, \quad a_6 + a_7 + a_8 = 0, \quad a_2 + a_5 + a_8 = 0$$
$$b_0 + b_1 + b_2 = \frac{1}{4}, \quad b_0 + b_3 + b_6 = 0, \quad b_6 + b_7 + b_8 = 0, \quad b_2 + b_5 + b_8 = \frac{1}{4}$$

- Subcase 4b:

$$a_0 + a_1 + a_2 = 0, a_0 + a_3 + a_6 = \frac{1}{4}, \quad a_6 + a_7 + a_8 = \frac{1}{4}, \quad a_2 + a_5 + a_8 = 0,$$
$$b_0 + b_1 + b_2 = 0, \quad b_0 + b_3 + b_6 = 0, \quad b_6 + b_7 + b_8 = \frac{1}{4}, \quad b_2 + b_5 + b_8 = \frac{1}{4}$$

- Subcase 4c:

$$a_0 + a_1 + a_2 = \frac{1}{4}, \quad a_0 + a_3 + a_6 = 0, \quad a_6 + a_7 + a_8 = 0, \quad a_2 + a_5 + a_8 = \frac{1}{4}$$
$$b_0 + b_1 + b_2 = \frac{1}{4}, \quad b_0 + b_3 + b_6 = \frac{1}{4}, \quad b_6 + b_7 + b_8 = 0, \quad b_2 + b_5 + b_8 = 0,$$

- Subcase 4d:

$$a_0 + a_1 + a_2 = 0, \quad a_0 + a_3 + a_6 = 0, \quad a_6 + a_7 + a_8 = \frac{1}{4}, \quad a_2 + a_5 + a_8 = \frac{1}{4}$$
$$b_0 + b_1 + b_2 = 0, \quad b_0 + b_3 + b_6 = \frac{1}{4}, \quad b_6 + b_7 + b_8 = \frac{1}{4}, \quad b_2 + b_5 + b_8 = 0.$$

We only study the Subcase 4a and leave the other three subcases to the interested reader. With the linear constraints, we tackle the nonlinear conditions (2)-(14). We use (2) and (3) to simplify (8) and (9) as follows:

$$\begin{aligned} 0 &= \sum_{\nu=a,b} \nu_0\nu_6 + \nu_1\nu_7 + \nu_2\nu_8 = \sum_{\nu=a,b} (\nu_0 + \nu_2)(\nu_6 + \nu_8) + \nu_1\nu_7 \\ &= \sum_{\nu=a,b} \left(\frac{1}{4} - \nu_1\right)(-\nu_7) + \nu_1\nu_7 = 2\sum_{\nu=a,b} (\nu_1\nu_7 - \frac{1}{8}\nu_7), \end{aligned} \tag{41}$$

and similarly,

$$\begin{aligned} 0 &= \sum_{\nu=a,b} (\nu_0\nu_2 + \nu_3\nu_5 + \nu_6\nu_8) = \sum_{\nu=a,b} (\nu_0 + \nu_6)(\nu_2 + \nu_8) + \nu_3\nu_5 \\ &= 2\left((\frac{1}{4} - a_3)(-a_5) + a_3a_5 + (-b_3)(\frac{1}{4} - b_5) + b_3b_5\right) \qquad (42) \\ &= 4\left(-\frac{a_5}{8} - \frac{b_3}{8} + a_3a_5 + b_3b_5\right). \qquad (43) \end{aligned}$$

As we did before, we have $a_4 = -\frac{1}{16} + \frac{1}{8\sqrt{2}}\cos\alpha$ and $b_4 = -\frac{1}{16} + \frac{1}{8\sqrt{2}}\sin\alpha$. It follows that

$$a_4^2 + b_4^2 + \frac{1}{8}(a_4 + b_4) = 0. \tag{44}$$

This fact will be used later.

With (8), i.e., (41), we can see that (12) holds. Indeed, the left-hand side of (12) is, by using (44),

$$\begin{aligned}
\sum_{\nu=a,b} (\nu_1(\nu_0+\nu_2) + \nu_4(\nu_3+\nu_5) + \nu_7(\nu_6+\nu_8)) \;&=\; \sum_{\nu=a,b} (\nu_1(\frac{1}{4} - \nu_1) - a_4^2 - a_7^2) \\
= \sum_{\nu=a,b} (\frac{1}{4}(-\nu_4 - \nu_7) - a_1^2 - a_7^2 - a_4^2) \;&=\; -\sum_{\nu=a,b} (\frac{1}{4}\nu_4 + (\nu_1+\nu_7)^2 + a_4^2) \\
= -\sum_{\nu=a,b} (\frac{1}{4}\nu_4 + 2\nu_4^2) \;=\; 0.
\end{aligned}$$

Similarly, we can show that with (9), i.e. (43), equation (13) holds.

If we add equations (10) and (11) together, we have by (44)

$$\begin{aligned}
&\sum_{\nu=a,b} \nu_4(\nu_0+\nu_2) + \nu_1(\nu_3+\nu_5) + \nu_7(\nu_3+\nu_5) + \nu_4(\nu_6+\nu_8) \\
&\quad = \sum_{\nu=a,b} (\nu_4(\frac{1}{4} - \nu_1) - \nu_1\nu_4 - \nu_7\nu_4 - \nu_4\nu_7) \\
&\quad = \sum_{\nu=a,b} \frac{1}{4}\nu_4 - 2\nu_4(\nu_1+\nu_7) \;=\; \sum_{\nu=a,b} \frac{1}{4}\nu_4 + 2\nu_4^2 \;=\; 0.
\end{aligned}$$

That is, the addition of (10) and (11) is always true. We only need to consider one of these two equations. Next we simplify equations (4) - (7).

Adding (4) and (8) with (3), we have

$$\begin{aligned}
0 \;&=\; \sum_{\nu=a,b} (\nu_2(\nu_7+\nu_8) + (\nu_0+\nu_1)\nu_6 + \nu_1\nu_7) \\
&=\; \sum_{\nu=a,b} (\frac{1}{4}\nu_6 + \nu_1\nu_7) \;=\; \sum_{\nu=a,b} \left(\frac{1}{4}\nu_6 + \frac{\nu_7}{8}\right).
\end{aligned} \tag{45}$$

Adding (5) and (8) together, we have

$$\begin{aligned}
0 \;&=\; \sum_{\nu=a,b} (\nu_1+\nu_2)\nu_8 + \nu_0(\nu_7+\nu_6) + \nu_1\nu_7 \\
&=\; \sum_{\nu=a,b} (\frac{1}{4}\nu_8 + \nu_1\nu_7) \;=\; \sum_{\nu=a,b} \left(\frac{1}{4}\nu_8 + \frac{1}{8}\nu_8\right).
\end{aligned} \tag{46}$$

It is easy to see that equations (45) and (46) are equivalent by using $a_6 + a_7 + a_8 = 0$ and $b_0 + b_1 + b_2 = 0$.

Adding (6) and (9) together, we have

$$0 \;=\; \sum_{\nu=a,b} (\nu_6(\nu_5+\nu_8) + \nu_2(\nu_0+\nu_3) + \nu_3\nu_5)$$

$$= \frac{1}{4}a_2 + a_3a_5 + \frac{1}{4}b_6 + b_3b_5 \ = \ \frac{1}{4}(a_2+b_6) + \frac{1}{8}(a_5+b_3). \tag{47}$$

Adding (7) and (9) together with (2), we have

$$\begin{aligned} 0 &= \sum_{\nu=a,b} (\nu_8(\nu_3+\nu_6) + \nu_0(\nu_2+\nu_5) + \nu_3\nu_5) \\ &= \frac{1}{4}a_8 + a_3a_5 + b_3b_5 + \frac{1}{4}b_0 \ = \ \frac{1}{4}(a_8+b_0) + \frac{1}{8}(a_5+b_3). \end{aligned} \tag{48}$$

In fact, equations (47) and (48) are equivalent by using $a_2+a_5+a_8=0$ and $b_0+b_3+b_6=0$.

Next we solve (41) using $a_1+a_4+a_7=0$ and $b_1+b_4+b_7=0$. Letting $p=-a_4$ and $q=-b_4$, we have $pa_7 - a_7^2 - \frac{a_7}{8} + qb_7 - b_7^2 - \frac{b_7}{8} = 0$. That is

$$(a_7 - \frac{p}{2} + \frac{1}{16})^2 + (b_7 - \frac{q}{2} + \frac{1}{16})^2 = \frac{1}{4}\left((p-\frac{1}{8})^2 + (q-\frac{1}{8})^2\right).$$

Recalling (44), i.e., $p^2 - \frac{1}{4}p + q^2 - \frac{1}{4}q = 0$, we have

$$\begin{aligned} (p-\frac{1}{8})^2 + (q-\frac{1}{8})^2 &= \frac{1}{8}(\frac{1}{4} - p - q) = \frac{1}{8}\left(\frac{1}{8} + \frac{1}{8\sqrt{2}}(\cos\alpha + \sin\alpha)\right) \\ &= \frac{1}{64}\left(1 + \frac{1}{\sqrt{2}}(\cos\alpha+\sin\alpha)\right) \ = \ \frac{1}{64}\left(1 + \sin(\alpha+\frac{\pi}{4})\right). \end{aligned}$$

Thus, we have

$$a_7 = \frac{p}{2} - \frac{1}{16} + \frac{1}{16}\sqrt{1+\sin(\alpha+\frac{\pi}{4})}\cos\beta, \quad b_7 = \frac{q}{2} - \frac{1}{16} + \frac{1}{16}\sqrt{1+\sin(\alpha+\frac{\pi}{4})}\sin\beta.$$

Similarly we can solve (43) using $a_3+a_4+a_5=0$ and $b_3+b_4+b_5=0$ giving us

$$a_5 = \frac{p}{2} - \frac{1}{16} + \frac{1}{16}\sqrt{1+\sin(\alpha+\frac{\pi}{4})}\cos\gamma, \quad b_3 = \frac{q}{2} - \frac{1}{16} + \frac{1}{16}\sqrt{1+\sin(\alpha+\frac{\pi}{4})}\sin\gamma.$$

It follows that

$$\begin{aligned} a_1 &= \tfrac{p}{2} - \tfrac{1}{16} + \tfrac{1}{16}\sqrt{1+\sin(\alpha+\tfrac{\pi}{4})}\cos\beta, \quad b_1 = \tfrac{q}{2} - \tfrac{1}{16} + \tfrac{1}{16}\sqrt{1+\sin(\alpha+\tfrac{\pi}{4})}\sin\beta \\ a_3 &= \tfrac{p}{2} + \tfrac{1}{16} + \tfrac{1}{16}\sqrt{1+\sin(\alpha+\tfrac{\pi}{4})}\cos\gamma, \quad b_5 = \tfrac{q}{2} + \tfrac{1}{16} + \tfrac{1}{16}\sqrt{1+\sin(\alpha+\tfrac{\pi}{4})}\sin\gamma. \end{aligned}$$

So, we still need to satisfy (2), (3), (46), (48), and (11). Using the linear relationships for Subcase 4a, (2) and (3) become

$$a_8^2 + (\frac{1}{4} - a_1 + a_5)a_8 + b_8^2(-b_1+b_5)b_8 \ = \ 0 \tag{49}$$

$$a_8^2 + (a_5+a_7)a_8 + b_8^2 + (b_5+b_7-\frac{1}{4})b_8 \ = \ -a_5a_7 - b_5b_7 + \frac{b_7}{4}. \tag{50}$$

After subtracting (49) from (50) and using $a_1+a_7=p$ and $b_1+b_7=q$, we can replace one of these equations with

$$(p-\frac{1}{4})a_8 + (q-\frac{1}{4})b_8 \ = \tfrac{1}{4}b_7 - a_5a_7 - b_5b_7. \tag{51}$$

Moreover, the linear relationships combined with (11) yield

$$pa_8 + qb_8 \;=\; \frac{1}{8}a_4 + \frac{1}{2}\sum_{\nu=a,b} \nu_1\nu_5 + \nu_3\nu_7 + (\nu_5 - \nu_1)\nu_4.$$

Thus, (46), (48), and (11) can be replaced by

$$a_8 + b_8 \;=\; -\frac{1}{2}(a_7 + b_7) \tag{52}$$

$$a_8 + b_8 \;=\; -\frac{1}{2}(a_5 - 2b_1 + b_3 + 2b_5) \tag{53}$$

$$a_8 + b_8 \;=\; -b_7 + \frac{1}{2}a_4 + 2\sum_{\nu=a,b} 2\nu_5\nu_7 + \nu_1\nu_5 + \nu_3\nu_7 + (\nu_5 - \nu_1)\nu_4. \tag{54}$$

Now, we only need to satisfy (49), (51), and (52)-(54).

Equating the right-hand sides of (52) and (53) gives

$$\sqrt{1 + \sin(\alpha + \frac{\pi}{4})}(\cos\gamma - \sin\gamma - \cos\beta + \sin\beta) = 0.$$

This constraint is satisfied whenever $\alpha = -\frac{3\pi}{4}, \gamma = \beta$, or $\gamma = -\beta - \frac{\pi}{2}$.

Equating the right hand sides of (52) and (54) gives

$$\begin{aligned} -\frac{1}{2}(a_7 + b_7) \;&=\; -b_7 + \frac{1}{2}a_4 + 2\sum_{\nu=a,b} (\nu_5(\nu_1 + \nu_7) + (\nu_3 + \nu_5)\nu_7 + \nu_4\nu_5 - \nu_1\nu_4 \\ &=\; -b_7 + \frac{1}{2}a_4 - 2\sum_{\nu=a,b} \nu_4(\nu_1 + \nu_7). \end{aligned}$$

Thus,

$$\begin{aligned} 0 \;&=\; \frac{1}{2}a_4 + 2\sum_{\nu=a,b} \nu_4^2 + \frac{\nu_4}{2} + \frac{\nu_7}{2} \\ &=\; 2(p^2 + q^2) - \frac{p}{4} - \frac{q}{4} + \frac{1}{32}\sqrt{1 + \sin(\alpha + \frac{\pi}{4})}(\cos\beta - \sin\beta) \\ &=\; \frac{1}{32}\sqrt{1 + \sin(\alpha + \frac{\pi}{4})}(\cos\beta - \sin\beta). \end{aligned}$$

Therefore, we only need to solve (49), (51), and (52) when $\alpha = -\frac{3\pi}{4}, \beta = \frac{\pi}{4}$, or $\beta = -\frac{3\pi}{4}$ and $\gamma = \beta$ or $\gamma = -\beta - \frac{\pi}{2}$.

We begin with $\alpha = -\frac{3\pi}{4}$, then

$$a_4 = b_4 = -\frac{1}{8}, \quad a_5 = a_7 = b_3 = b_7 = 0, \quad a_1 = a_3 = b_1 = b_5 = \frac{1}{8}$$

which reduces (49), (51), and (52) to

$$(a_8 + \frac{1}{8})a_8 + b_8^2 = 0, \quad a_8 + b_8 = 0.$$

These equations yield two solutions: $a_8 = b_8 = 0$ or $a_8 = -\frac{1}{16}$ and $b_8 = \frac{1}{16}$.

Now, with $\beta = \gamma = \frac{\pi}{4}$, we solve for a_8 and b_8 using the linear equations (51) and (52), we have

$$a_8 = -\frac{p}{4} - \frac{\sqrt{2}}{8}\sqrt{1 + \sin(\alpha + \pi/4)}), \quad b_8 = -\frac{q}{4} + \frac{1}{16}.$$

Plugging these solutions into (49) and simplifying we have

$$\cos(\alpha - \frac{\pi}{4}) - 1 = 4\sqrt{2 + 2\cos(\alpha - \frac{\pi}{4})}$$

which has no real solution.

Similarly for $\beta = \gamma = -\frac{3\pi}{4}$, (51) and (52) yield

$$a_8 = -\frac{p}{4} + \frac{\sqrt{2}}{8}\sqrt{1 + \sin(\alpha + \pi/4)}), \quad b_8 = -\frac{q}{4} + \frac{1}{16},$$

but now (49) reduces to

$$\cos(\alpha - \frac{\pi}{4}) - 1 = -4\sqrt{2 + 2\cos(\alpha - \frac{\pi}{4})}$$

which has two solutions: $\alpha = \frac{\pi}{4} \pm \cos^{-1}(17 - 8\sqrt{5})$.

Finally for $\beta = \frac{\pi}{4}$ and $\gamma = -\frac{3\pi}{4}$, the linear equations produce

$$a_8 = -\frac{p}{4}, \quad b_8 = -\frac{q}{4} + \frac{1}{16} - \frac{\sqrt{2}}{32}\sqrt{1 + \sin(\alpha - \frac{\pi}{4})},$$

and similarly for $\beta = -\frac{3\pi}{4}$ and $\gamma = \frac{\pi}{4}$

$$a_8 = -\frac{p}{4}, \quad b_8 = -\frac{q}{4} + \frac{1}{16} + \frac{\sqrt{2}}{32}\sqrt{1 + \sin(\alpha - \frac{\pi}{4})}.$$

Both of these choices for a_8 and b_8 after being substituted into (49) require

$$p^2 + q^2 = 0$$

which is satisfied by $\alpha = \frac{\pi}{4}$.

Therefore, the complete solution for Subcase 4a has 6 solitary solutions. The four rational solutions are given below:

$$\frac{1}{8}\begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 & 1 \\ 1 & 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}, \quad \frac{1}{16}\begin{bmatrix} 1 & 1 & 2 & 2 & 1 & 1 \\ 1 & -1 & 0 & 0 & -1 & 1 \\ 2 & 0 & -2 & -2 & 0 & 2 \\ 2 & 0 & -2 & -2 & 0 & 2 \\ 1 & -1 & 0 & 0 & -1 & 1 \\ 1 & 1 & 2 & 2 & 1 & 1 \end{bmatrix},$$

$$\frac{1}{8}\begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}, \quad \frac{1}{8}\begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & -1 & 1 \\ 1 & -1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

3. ORTHOGONALITY

In this section, we discuss the orthogonality of the solutions from Cases 1-4. We begin with a review of the Lawton condition as well as a well known necessary and sufficient condition for orthogonality. We conclude this section with a numerical experiment consisting of a one-level decomposition of two gray-scale images using a nonseparable filter from Subcase 4a, and compare its performance with Haar and D4.

Let

$$m(e^{i\omega_1}, e^{i\omega_2}) = \sum_{k,\ell} h_{k,l} e^{ik\omega_1} e^{i\ell\omega_2},$$

where the $h_{k,\ell}$'s are the a_i's and b_i's as discussed previously in Section 2. Define

$$\hat{\phi}(\omega_1, \omega_2) \;=\; \prod_{k=1}^{\infty} m\left(e^{\frac{i\omega_1}{2^k}}, e^{\frac{i\omega_2}{2^k}}\right). \tag{55}$$

For the coefficients as defined in Section 2, we know that ϕ is well defined and is in $L_2(\mathbf{R}^2)$. Define

$$\alpha_{\ell_1,\ell_2} \;=\; \int_{\mathbf{R}^2} \phi(x,y)\overline{\phi(x-\ell_1, y-\ell_2)}\mathrm{d}x\mathrm{d}y. \tag{56}$$

Thus, if $\alpha_{\ell_1,\ell_2} = \delta_{\ell_1,\ell_2}$, then ϕ is orthonormal. By the refinement equation

$$\phi(x,y) = 4 \sum_{k_1,k_2=0}^{5} h_{k_1,k_2}\phi(2x-k_1, 2y-k_2). \tag{57}$$

Using (57) in (56) we have

$$\alpha_{\ell_1,\ell_2} \;=\; 4\sum_{n_1,n_2}\left(\sum_{k_1,k_2} h_{k_1,k_2} h_{k_1+n_1-2\ell_1, k_2+n_2-2\ell_2}\right)\alpha_{n_1,n_2}. \tag{58}$$

Because the $supp(\phi) \subset [0,5]^2$, the only possible nonzero α_{ℓ_1,ℓ_2} are for $-4 \le \ell_1, \ell_2 \le 4$. Let α be the vector of length 81 consisting of the α_{ℓ_1,ℓ_2}'s for some fixed ordering of the indices in the range of $-4 \le \ell_1, \ell_2 \le 4$ and define the matrix

$$A_{(\ell_1,\ell_2),(n_1,n_2)} = 4\sum_{k_1,k_2} h_{k_1,k_2} h_{k_1+n_1-2\ell_1, k_2+n_2-2\ell_2} \tag{59}$$

for this same ordering. Then equation (58) says that α is an eigenvector of A with eigenvalue$\lambda = 1$, i.e., $\alpha = A\alpha$. Now, condition (i) of Section 1 implies that

$$4\sum_{k_1,k_2} h_{k_1,k_2} h_{k_1-2j_1, k_2-2j_2} x^{2j_1} y^{2j_2} = 1,$$

i.e.

$$4\sum_{k_1,k_2} h_{k_1,k_2} h_{k_1-2j_1, k_2-2j_2} = \delta_{j_1,j_2}.$$

Thus the vector δ of length 81 consisting of the entries δ_{ℓ_1,ℓ_2} for the same ordering as before is also an eigenvector for A with eigenvalue $\lambda = 1$. For completeness, we state the generalization of Lawton's condition (cf. [10]) in R^2.

THEOREM 3.1. *Let $m(x,y)$ be a given polynomial satisfying (i) and (ii) and A a matrix defined as in equation (59) for the coefficients of $m(x,y)$. Let ϕ be the function generated by equation (55). If $\lambda = 1$ is a non-degenerate eigenvalue of A, then $\{\phi(\cdot - \ell_1, \cdot - \ell_2) | (\ell_1, \ell_2) \in \mathbf{Z}^2\}$ is an orthonormal set.*

We also need the following well-known necessary and sufficient condition for orthonormality.

THEOREM 3.2. *Let $m(x,y)$ be a given polynomial satisfying (i) and (ii). Let ϕ be the function generated by equation (55). Then $\{\phi(\cdot - \ell_1, \cdot - \ell_2) | (\ell_1, \ell_2) \in \mathbf{Z}^2\}$ is an orthonormal set if and only if*

$$\sum_{(k,\ell)\in\mathbf{Z}^2} |\widehat{\phi}((\omega_1,\omega_2) + 2\pi(k,\ell))|^2 = 1, \quad \forall(\omega_1,\omega_2) \in [-\pi,\pi]^2.$$

.

Case 1: We have used Matlab and Mathematica to check the eigenvalues of the Lawton matrix associated with this two-parameter family for a large sample of parameters. The eigenvalue $\lambda = 1$ was non-degenerate for every sample we tested.

Case 2: These solutions are not associated with scaling functions. The conditions for Case 2a immediately imply $m(e^{i\omega_1}, 1) = (1+e^{5i\omega_1})/2$. If we consider the one-dimensional restriction $\bar{m}(\omega) := m(e^{i\omega}, 1)$, we see that $\{\pm\frac{\pi}{5}, \pm\frac{3\pi}{5}, \pm\pi\}$ are the zeros of $\bar{m}(\omega)$. Because $m(e^{i\omega}, -1) = 0$, condition (ii) implies that $|\bar{m}(\omega)|^2 + |\bar{m}(\omega+\pi)|^2 = 1$. Moreover,

$$\left|\bar{m}\left(-\frac{3\pi}{5}+\pi\right)\right| = \left|\bar{m}\left(-\frac{\pi}{5}+\pi\right)\right| = \left|\bar{m}\left(\frac{3\pi}{5}-\pi\right)\right| = \left|\bar{m}\left(\frac{\pi}{5}-\pi\right)\right| = 1.$$

Because $\left\{\xi_1 = \frac{2\pi}{5}, \xi_2 = \frac{4\pi}{5}, \xi_3 = -\frac{2\pi}{5}, \xi_4 = -\frac{4\pi}{5}\right\}$ is a nontrivial cycle in $[-\pi,\pi]$ for the operation $\xi \to 2\xi \bmod 2\pi$ such that $|\bar{m}(\xi_i)| = 1$, the set of functions $\{\bar{\phi}(\cdot - n)\}_{n\in\mathbf{Z}}$ associated with $\bar{m}(\omega)$ is not orthonormal and

$$\sum_k |\hat{\bar{\phi}}(\frac{2\pi}{5} + 2\pi k)|^2 = 0$$

(See [5]). So, for the unrestricted function, we have

$$\sum_{k,l} \left|\hat{\phi}\left(\frac{2\pi}{5} + 2\pi k, 2\pi\ell\right)\right|^2 = 0$$

which contradicts Theorem 6.2.

The other subcase 2b has the property that $m(e^{i\omega_1}, 1) = (1 + e^{i3\omega_1})/2$ which similarly excludes it from being associated with scaling functions.

Case 3: This case has the same problematic factors as Case 2 but with respect to the other component $m(1, e^{i\omega_2}) = (1 + e^{i5\omega_2})/2$ or $m(1, e^{i\omega_2}) = (1 + e^{i3\omega_2})/2$.

Case 4: The solutions for this final case have the same factors as in Cases 2 and 3.

So, Case 1 is the only solution associated with scaling functions since the other cases failed to satisfy the necessary and sufficient condition for othogonality. Although, the refinable functions for Cases 2-4 are not orthogonal to their shifts they still have associated tight frames since they satisfy condition (ii). These cases are analogous to the univariate Haar function with support $[0, 3]$.

4. THE 8×8 CASE

In this section, We derive several necessary conditions from the properties (i)-(iv) with $N = 7$ and $M = 1$. We will use these necessary conditions to show in the next section that the second order vanishing moment $M = 2$ is not possible for this support size.

Let $N = 7$ and consider $m(x, y) = \sum_{i=0}^{7}\sum_{j=0}^{7} c_{ij}x^i y^j$ which satisfies properties (i)-(iv) with $M = 1$. The symmetry property (iii) implies that

$$m(x,y) = \begin{bmatrix} 1 \\ y \\ y^2 \\ y^3 \\ y^4 \\ y^5 \\ y^6 \\ y^7 \end{bmatrix}^T \begin{bmatrix} a_0 & b_0 & a_1 & b_1 & a_2 & b_2 & a_3 & b_3 \\ b_{15} & a_{15} & b_{14} & a_{14} & b_{13} & a_{13} & b_{12} & a_{12} \\ a_4 & b_4 & a_5 & b_5 & a_6 & b_6 & a_7 & b_7 \\ b_{11} & a_{11} & b_{10} & a_{10} & b_9 & a_9 & b_8 & a_8 \\ a_8 & b_8 & a_9 & b_9 & a_{10} & b_{10} & a_{11} & b_{11} \\ b_7 & a_7 & b_6 & a_6 & b_5 & a_5 & b_4 & a_4 \\ a_{12} & b_{12} & a_{13} & b_{13} & a_{14} & b_{14} & a_{15} & b_{15} \\ b_3 & a_3 & b_2 & a_2 & b_1 & a_1 & b_0 & a_0 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \\ x^3 \\ x^4 \\ x^5 \\ x^6 \\ x^7 \end{bmatrix}. \tag{60}$$

Properties (i) and (iii) imply that

$$\sum_{i=0}^{15}\sum_{\nu=a,b} \nu_i = \frac{1}{2}. \tag{61}$$

Property (ii) implies the following 25 nonlinear equations:

$$\sum_{\nu=a,b} \nu_0\nu_{15} = 0 \tag{62}$$

$$\sum_{\nu=a,b} \nu_3\nu_{12} = 0 \tag{63}$$

$$\sum_{\nu=a,b} \nu_0\nu_{11} + \nu_4\nu_{15} = 0 \tag{64}$$

$$\sum_{\nu=a,b} \nu_0\nu_{14} + \nu_1\nu_{15} = 0 \tag{65}$$

$$\sum_{\nu=a,b} \nu_2\nu_{12} + \nu_3\nu_{13} = 0 \tag{66}$$

$$\sum_{\nu=a,b} \nu_3\nu_8 + \nu_7\nu_{12} = 0 \tag{67}$$

$$\sum_{\nu=a,b} \nu_0\nu_7 + \nu_4\nu_{11} + \nu_8\nu_{15} = 0 \tag{68}$$

$$\sum_{\nu=a,b} \nu_0\nu_{13} + \nu_1\nu_{14} + \nu_2\nu_{15} = 0 \tag{69}$$

$$\sum_{\nu=a,b} \nu_1\nu_{12} + \nu_2\nu_{13} + \nu_3\nu_{14} = 0 \tag{70}$$

$$\sum_{\nu=a,b} \nu_3\nu_4 + \nu_7\nu_8 + \nu_{11}\nu_{12} = 0 \tag{71}$$

$$\sum_{\nu=a,b} \nu_0\nu_3 + \nu_4\nu_7 + \nu_8\nu_{11} + \nu_{12}\nu_{15} = 0 \tag{72}$$

$$\sum_{\nu=a,b} \nu_0\nu_{10} + \nu_1\nu_{11} + \nu_4\nu_{14} + \nu_5\nu_{15} = 0 \tag{73}$$

$$\sum_{\nu=a,b} \nu_0\nu_{12} + \nu_1\nu_{13} + \nu_2\nu_{14} + \nu_3\nu_{15} = 0 \tag{74}$$

$$\sum_{\nu=a,b} \nu_2\nu_8 + \nu_3\nu_9 + \nu_6\nu_{12} + \nu_7\nu_{13} = 0 \tag{75}$$

$$\sum_{\nu=a,b} \nu_0\nu_6 + \nu_1\nu_7 + \nu_4\nu_{10} + \nu_5\nu_{11} + \nu_8\nu_{14} + \nu_9\nu_{15} = 0 \tag{76}$$

$$\sum_{\nu=a,b} \nu_0\nu_9 + \nu_1\nu_{10} + \nu_2\nu_{11} + \nu_4\nu_{13} + \nu_5\nu_{14} + \nu_6\nu_{15} = 0 \tag{77}$$

$$\sum_{\nu=a,b} \nu_1\nu_8 + \nu_2\nu_9 + \nu_3\nu_{10} + \nu_5\nu_{12} + \nu_6\nu_{13} + \nu_7\nu_{14} = 0 \tag{78}$$

$$\sum_{\nu=a,b} \nu_2\nu_4 + \nu_3\nu_5 + \nu_6\nu_8 + \nu_7\nu_9 + \nu_{10}\nu_{12} + \nu_{11}\nu_{13} = 0 \tag{79}$$

$$\sum_{\nu=a,b} \nu_0\nu_8 + \nu_1\nu_9 + \nu_2\nu_{10} + \nu_3\nu_{11} + \nu_4\nu_{12} + \nu_5\nu_{13} + \nu_6\nu_{14} + \nu_7\nu_{15} = 0 \tag{80}$$

$$\sum_{\nu=a,b} \nu_0\nu_2 + \nu_1\nu_3 + \nu_4\nu_6 + \nu_5\nu_7 + \nu_8\nu_{10} + \nu_9\nu_{11} + \nu_{12}\nu_{14} + \nu_{13}\nu_{15} = 0 \tag{81}$$

$$\sum_{\nu=a,b} \nu_0\nu_5 + \nu_1\nu_6 + \nu_2\nu_7 + \nu_4\nu_9 + \nu_5\nu_{10} + \nu_6\nu_{11} + \nu_8\nu_{13} + \nu_9\nu_{14} + \nu_{10}\nu_{15} = 0 \tag{82}$$

$$\sum_{\nu=a,b} \nu_1\nu_4 + \nu_2\nu_5 + \nu_3\nu_6 + \nu_5\nu_8 + \nu_6\nu_9 + \nu_7\nu_{10} + \nu_9\nu_{12} + \nu_{10}\nu_{13} + \nu_{11}\nu_{14} = 0 \tag{83}$$

$$\sum_{\nu=a,b} \nu_0\nu_1 + \nu_1\nu_2 + \nu_2\nu_3 + \nu_4\nu_5 + \nu_5\nu_6 + \nu_6\nu_7 + \nu_8\nu_9 + \nu_9\nu_{10} + \nu_{10}\nu_{11} + \nu_{12}\nu_{13} + \nu_{13}\nu_{14} + \nu_{14}\nu_{15} = 0 \tag{84}$$

$$\sum_{\nu=a,b} \nu_0\nu_4 + \nu_1\nu_5 + \nu_2\nu_6 + \nu_3\nu_7 + \nu_4\nu_8 + \nu_5\nu_9 + \nu_6\nu_{10} + \nu_7\nu_{11} + \nu_8\nu_{12} + \nu_9\nu_{13} + \nu_{10}\nu_{14} + \nu_{11}\nu_{15} = 0 \tag{85}$$

$$\sum_{i=0}^{15} \sum_{\nu=a,b} \nu_i^2 = \frac{1}{8}. \tag{86}$$

Property (iv) with $M = 1$ implies

$$\begin{aligned}
a_0 + a_1 + a_2 + a_3 &= b_0 + b_1 + b_2 + b_3 && (87)\\
a_4 + a_5 + a_6 + a_7 &= b_4 + b_5 + b_6 + b_7 && (88)\\
a_8 + a_9 + a_{10} + a_{11} &= b_8 + b_9 + b_{10} + b_{11} && (89)\\
a_{12} + a_{13} + a_{14} + a_{15} &= b_{12} + b_{13} + b_{14} + b_{15} && (90)\\
a_0 + a_4 + a_8 + a_{12} &= b_3 + b_7 + b_{11} + b_{15} && (91)\\
a_1 + a_5 + a_9 + a_{13} &= b_2 + b_6 + b_{10} + b_{14} && (92)\\
a_2 + a_6 + a_{10} + a_{14} &= b_1 + b_5 + b_9 + b_{13} && (93)\\
a_3 + a_7 + a_{11} + a_{15} &= b_0 + b_4 + b_8 + b_{12}. && (94)
\end{aligned}$$

Using (87)-(90) and (61), we immediately have

$$\sum_{i=0}^{15} a_i = \sum_{i=0}^{15} b_i = \frac{1}{4}. \tag{95}$$

Next, we use various combinations of the nonlinear equations to make perfect squares as we have done previously. This enables us to introduce parameters in order to simplify these equations.

Using (86), (85), (80), and (74), we have

$$\sum_{\nu=a,b} (\nu_0 + \nu_4 + \nu_8 + \nu_{12})^2 + (\nu_1 + \nu_5 + \nu_9 + \nu_{13})^2 + (\nu_2 + \nu_6 + \nu_{10} + \nu_{14})^2 + (\nu_3 + \nu_7 + \nu_{11} + \nu_{15})^2 = \frac{1}{8}. \tag{96}$$

Moreover, using (65), (66), (73), (75), (76), (79), (81), we have

$$\sum_{\nu=a,b} (\nu_0 + \nu_4 + \nu_8 + \nu_{12})(\nu_2 + \nu_6 + \nu_{10} + \nu_{14}) + (\nu_1 + \nu_5 + \nu_9 + \nu_{13})(\nu_3 + \nu_7 + \nu_{11} + \nu_{15}) = 0. \tag{97}$$

After using (91)-(94), (96), and (97), we obtain

$$((a_0 + a_4 + a_8 + a_{12}) \pm (a_2 + a_6 + a_{10} + a_{14}))^2 + ((a_1 + a_5 + a_9 + a_{13}) \pm (a_3 + a_7 + a_{11} + a_{15}))^2 = \frac{1}{8}. \tag{98}$$

Choosing the plus sign in equation (98) and using (95) yields

$$a_0 + a_4 + a_8 + a_{12} + a_2 + a_6 + a_{10} + a_{14} = r_0, \tag{99}$$
$$a_1 + a_5 + a_9 + a_{13} + a_3 + a_7 + a_{11} + a_{15} = s_0, \tag{100}$$

where $r_0 + s_0 = 1/4$ and $r_0 s_0 = 0$. Giving us four cases: $r_0, s_0 = 1/4$ or 0.

Similarly, using (86), (84), (81), and (72), we have

$$\sum_{\nu=a,b} (\nu_0+\nu_1+\nu_2+\nu_3)^2 + (\nu_4+\nu_5+\nu_6+\nu_7)^2 + (\nu_8+\nu_9+\nu_{10}+\nu_{11})^2 + (\nu_{12}+\nu_{13}+\nu_{14}+\nu_{15})^2 = \frac{1}{8}. \tag{101}$$

Using (80), (78), (77), (75), (73), (67), (64), we have

$$\sum_{\nu=a,b} (\nu_0+\nu_1+\nu_2+\nu_3)(\nu_8+\nu_9+\nu_{10}+\nu_{11}) + (\nu_4+\nu_5+\nu_6+\nu_7)(\nu_{12}+\nu_{13}+\nu_{14}+\nu_{15}) = 0. \tag{102}$$

In a similar fashion as before, we have

$$a_0 + a_1 + a_2 + a_3 + a_8 + a_9 + a_{10} + a_{11} = t_0, \tag{103}$$
$$a_4 + a_5 + a_6 + a_7 + a_{12} + a_{13} + a_{14} + a_{15} = u_0, \tag{104}$$

where $t_0 + u_0 = 1/4$ and $t_0 u_0 = 0$. We now refine our solutions to sums of four coefficients. Choosing the minus sign in equation (98) yields

$$a_0 + a_4 + a_8 + a_{12} = \frac{1}{2}(r_0 + r_1), \quad a_2 + a_6 + a_{10} + a_{14} = \frac{1}{2}(r_0 - r_1)$$
$$a_1 + a_5 + a_9 + a_{13} = \frac{1}{2}(s_0 + s_1), \quad a_3 + a_7 + a_{11} + a_{15} = \frac{1}{2}(s_0 - s_1),$$

where $r_1 = \frac{1}{4}\cos\alpha$ and $s_1 = \frac{1}{4}\sin\alpha$. Furthermore, equations (62), (63), (64), (67), (68), (71), and (72) yield

$$\sum (a_0 + a_4 + a_8 + a_{12})(a_3 + a_7 + a_{11} + a_{15}) = 0.$$

This together with (91) and (94), give the following constraint on our parameters

$$2(r_0 + r_1)(s_0 - s_1) = 0. \tag{105}$$

Because of the relationship between r_1 and s_1, equation (105) produces three cases: $r_1 = r_0$ and $s_1 = s_0$, $r_1 = -r_0$ and $s_1 = s_0$, or $r_1 = -r_0$ and $s_1 = -s_0$.

Similarly,

$$a_0 + a_1 + a_2 + a_3 = \frac{1}{2}(t_0 + t_1), \quad a_8 + a_9 + a_{10} + a_{11} = \frac{1}{2}(t_0 - t_1)$$
$$a_4 + a_5 + a_6 + a_7 = \frac{1}{2}(u_0 + u_1), \quad a_{12} + a_{13} + a_{14} + a_{15} = \frac{1}{2}(u_0 - u_1),$$

where $t_1 = \frac{1}{4}\cos\beta$ and $u_1 = \frac{1}{4}\sin\beta$. Additionally, equations (62), (63), (65), (69), (70), and (74) with (87) and (90), gives us $(t_0 + t_1)(u_0 - u_1) = 0$. In summary, we have the following lemma

LEMMA 4.1.

$$\begin{aligned}
a_0+a_4+a_8+a_{12} &= \tfrac{1}{2}(r_0+r_1), & b_0+b_4+b_8+b_{12} &= \tfrac{1}{2}(s_0-s_1),\\
a_1+a_5+a_9+a_{13} &= \tfrac{1}{2}(s_0+s_1), & b_1+b_5+b_9+b_{13} &= \tfrac{1}{2}(r_0-r_1)\\
a_2+a_6+a_{10}+a_{14} &= \tfrac{1}{2}(r_0-r_1), & b_2+b_6+b_{10}+b_{14} &= \tfrac{1}{2}(s_0+s_1),\\
a_3+a_7+a_{11}+a_{15} &= \tfrac{1}{2}(s_0-s_1), & b_3+b_7+b_{11}+b_{15} &= \tfrac{1}{2}(r_0+r_1),\\
a_0+a_1+a_2+a_3 &= \tfrac{1}{2}(t_0+t_1), & b_0+b_1+b_2+b_3 &= \tfrac{1}{2}(t_0+t_1),\\
a_4+a_5+a_6+a_7 &= \tfrac{1}{2}(u_0+u_1), & b_4+b_5+b_6+b_7 &= \tfrac{1}{2}(u_0+u_1),\\
a_8+a_9+a_{10}+a_{11} &= \tfrac{1}{2}(t_0-t_1), & b_8+b_9+b_{10}+b_{11} &= \tfrac{1}{2}(t_0-t_1),\\
a_{12}+a_{13}+a_{14}+a_{15} &= \tfrac{1}{2}(u_0-u_1), & b_{12}+b_{13}+b_{14}+b_{15} &= \tfrac{1}{2}(u_0-u_1).
\end{aligned}$$

where

$$r_0+s_0=\frac{1}{4},\ \ r_0s_0=0,\ \ (r_0+r_1)(s_0-s_1)=0,\ \ r_1=\frac{1}{4}\cos\alpha,\ \ s_1 \ = \ \frac{1}{4}\sin\alpha,$$
$$t_0+u_0=\frac{1}{4},\ \ t_0u_0=0,\ \ (t_0+t_1)(u_0-u_1)=0,\ \ t_1=\frac{1}{4}\cos\beta,\ \ u_1 \ = \ \frac{1}{4}\sin\beta.$$

Because there are no symmetric compactly-supported tensor-product wavelets with more than one vanishing moment, it is natural to ask whether there are any nonseparable symmetric solutions with multiple vanishing moments. In [9], it is shown that their are no symmetric solutions with higher vanishing moments for $m(x,y)$ with $N=7$. Although the bivariate case allows enough freedom to generate a family of symmetric solutions, it does not allow for multiple vanishing moments at least for the support size we have considered. So, compact support, orthogonality, vanishing moments, and symmetry are again at odds in the construction of bivariate wavelets.

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